On the Efficiency of the Walrasian Mechanism

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First Welfare Theorem

- If there exist prices in the market such that no good is under- or over-demanded, then those prices implement an efficient allocation.

- Given some natural conditions (*gross substitutability*), such prices always exist.

- Those prices can be found via very natural distributed greedy algorithms.

First Welfare Theorem

- Set of agent $N = \{1...n\}$ and goods $M = \{1...m\}$
- Each agent has a valuation $v_i : 2^M \to \mathbb{R}_+$
- Demands: for prices $p \in \mathbb{R}_+^M$, each agent purchases his favorite bundle: $D(v_i;p) = \text{argmax}_S[v_i(S) - \sum_{j \in S} p_j]$
- Walrasian prices $p \in \mathbb{R}_+^M$ s.t. there exist $S_i \in D(v_i;p)$ that clear the market.
- Efficiency: in a WE, the welfare $\sum_i v_i(S_i)$ is maximized.

First Welfare Theorem

- [Kelso-Crawford] If the valuations are *gross substitutes*, a Walrasian equilibrium always exists and can be found via *tatonnement* (trial-and-error).
  - Fix arbitrary prices: $p \in \mathbb{R}_+^M$
  - Compute demands $S_i \in D(v_i; p)$
  - For every $j \in N$ if
    - j is not demanded $p_j \leftarrow p_j \cdot (1 - \epsilon)$
    - j is over-demanded $p_j \leftarrow p_j \cdot (1 + \epsilon)$

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  - Fix arbitrary prices: $p \in \mathbb{R}_+^M$
  - $\boxed{\text{Compute demands } S_i \in D(v_i; p)}$ truthfully reporting preferences
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$\mathbb{R}^M_+$

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Large vs Small Markets

Large vs Small Markets

Price taking behavior

Bargaining / Haggling

Strategic demand

Our goal: prove welfare theorems with strategic agents

Related Work

- [Hurwicz'72]: observes that market equilibrium is not strategy-proof and proposes a *game-theoretic framework* to analyze its equilibrium properties.

- [Rutischi, Sattherwaite, Williams, Econometrica'94] [Sattherwaite, Williams, Econometrica'02]: observe that many markets use variations of market clearing, such as stock exchange opening price or call market for copper and gold and observe: *"Such behavior, which is the essence of bargaining, may lead to an impasse that delays or lesses the gains of trade"*

Related Work

- [Jackson, Manelli], [Otani, Sicilian], [Roberts, Postlewaite], [Azevedo, Budish]: for **large** markets and suitable regularity conditions, the Walrasian mechanism is **approximately** strategyproof.

- **Here**: Approximate version of the first welfare theorem without any large market or regularity assumptions.

Related Work

- Our perspective: PoA of Auctions [Christodoulou, Kovacs and Shapira] and follow up work…

- Also on strategic aspects of markets: [Markakis, Telelis], [de Keijze, Markakis, Shafer, Telelis], [Adsul, Babu, Garg, Mehta, Sohoni], [Chen, Deng, Zhang, Zhang], [Zhang], …

Hurwicz Framework

- Each agent has a valuation $v_i : 2^M \to \mathbb{R}_+$
- ... but reports $b_i : 2^M \to \mathbb{R}_+$ (bid)
- compute allocation and prices according to a Walrasian equilibrium $p \in \mathbb{R}_+^M, \{S_i\}_i$ of the *reported* market.
- Utilities: $u_i = v_i(S_i) - \sum_{j \in S_i} p_j$
- Welfare: $W = \sum_i v_i(S_i)$

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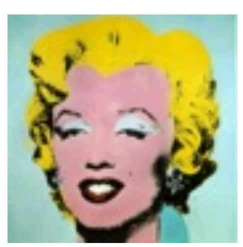
Example

			
	1.1	1.1	1.1
	2	2	4

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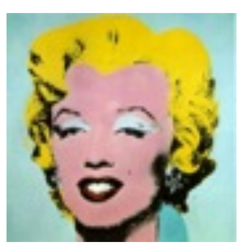	2	2	4
prices	1.1	1.1	u = 1.8

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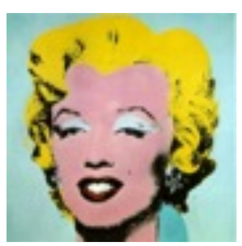	~~2~~ 2	~~2~~ 0	~~4~~ 2

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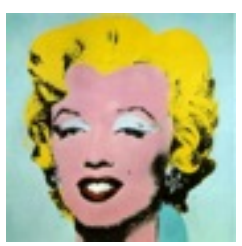	~~2~~ 2	~~2~~ 0	~~4~~ 2
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Assumption: γ- exposure: $\text{Payment}(b_i, b_{-i}) \leq (1+\gamma)v_i(S_i), \forall b_{-i}$

Main Theorem: If agent values and bids are Gross substitutes and agents employ γ-exposure strategies, then for all Nash equilibria of (any flavor of) the Walrasian mechanism:

$$\sum_i v_i(S_i) \geq \frac{1}{4+2\gamma} \sum_i v_i(S_i^*)$$

- guarantees also hold for the (correlated) Bayesian setting
- existence of efficient pure 0-exposure equilibria (PoS = 1)
- lower bound of **2** for 0-exposure

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Examples: Walrasian mechanism, VCG, Pay-Your-Bid, ...

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General Theorem: If agent values are in $\mathcal{V}$, bids are in $\mathcal{B}$, and agents employ γ-exposure strategies, then the Price of Anarchy of mechanism $\mathcal{M}$ is PoA .

$\mathcal{M}$	$\mathcal{B}$	$\mathcal{V}$	PoA
declared welfare maximizers	GS	GS	$4 + 2\gamma$

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So far, we considered $\mathcal{B} = \mathcal{V}$. However, a simpler bidding language can be useful for various reasons:

- representation / communication
- computational efficiency
- auction simplicity

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e.g. item bidding auctions [CKS], [BR], [FFGL].

We can allow for more expressive, yet still computationally efficient mechanisms, i.e., run the Walrasian mechanism with GS bids, even if valuations are XOS.

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valuation compression in VCG [Dutting, Henzinger, Starnberger]

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Pay-Your-Bid	GS	GS	2
Pay-Your-Bid	XOS	XOS	3

Conclusion

- Efficiency of market equilibrium with strategic agents without large market assumptions
- Unified efficiency guarantees for various auctions: Walrasian mechanism, item bidding auctions, VCG with restricted bidding language, pay-your-bid auctions, ...

Open Problems

- Efficiency guarantees for **approximately** welfare maximizers, i.e., [Lehmann, Lehmann, Nisan], [Fu, Kleinberg, Lavi]

- Efficiency guarantees in more sophisticated markets : buyers/sellers, budgets, ...

- Matching bounds for the Price of Anarchy for the Walrasian mechanism ? Right now, lower bound $= 2$ and upper bound $= 4$.