Computing Walrasian Equilibrium

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supplies:
flour,
milk,
vegetables,
medicine,
paper,
...

demand:
bakeries,
hospitals,
households,
schools,
...

supplies:

flour,

milk,

vegetables,

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...

Task: Allocate supplies efficiently to satisfy the demands of the city.

demand:

bakeries,

hospitals,

households,

schools,

...

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flour,
milk,
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medicine,
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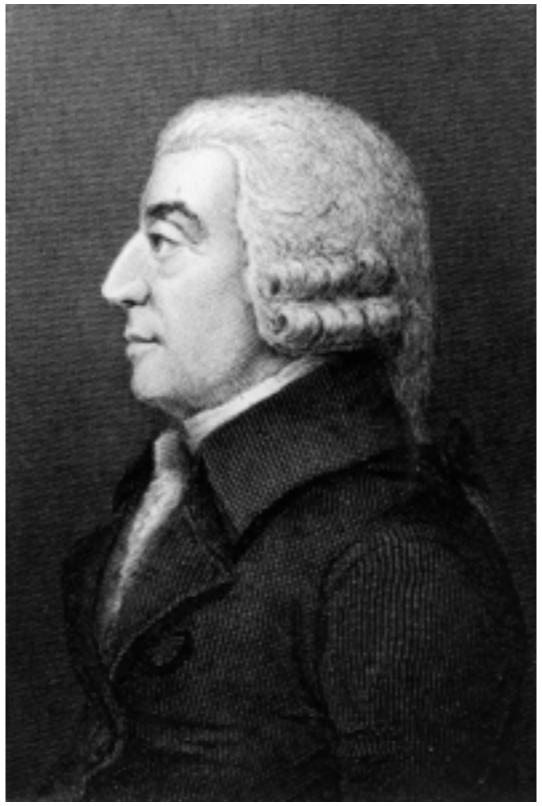
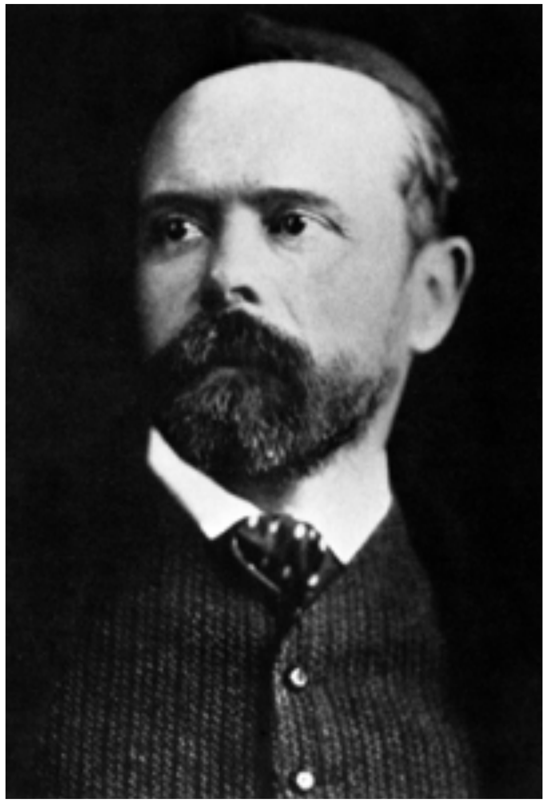
Invisible Hand
of the market

Theory of Market Equilibrium



- Adam Smith: "Wealth of the Nations" (1776): invisible hand

- Leon Walras: "Elements of Pure Economics" (1874): mathematical theory of market equilibrium

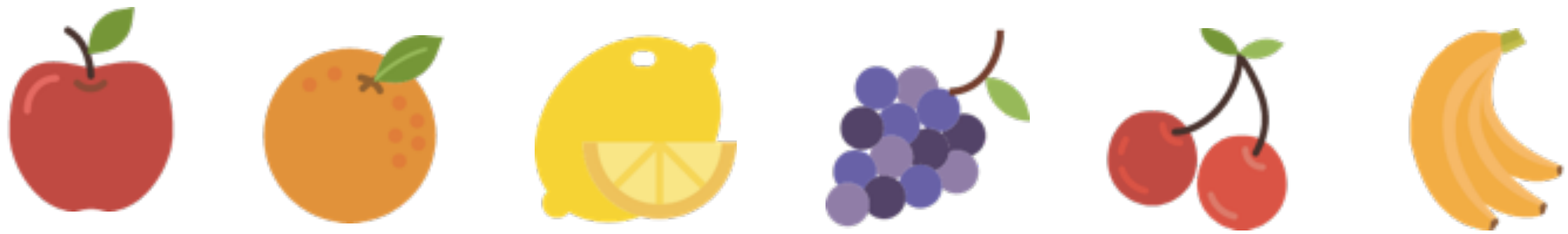


- Arrow-Debreu (1950's): general equilibrium theory

- Kelso-Crawford (1982): discrete and combinatorial theory of market equilibr.

Market equilibrium

n goods



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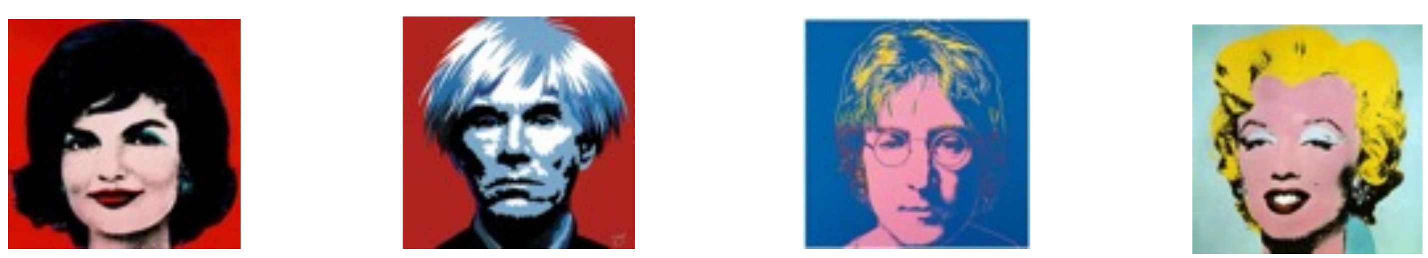

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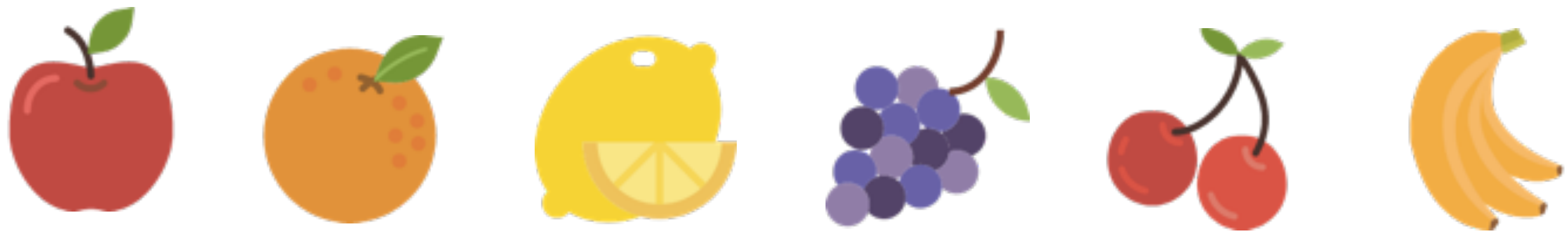

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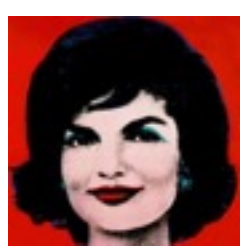
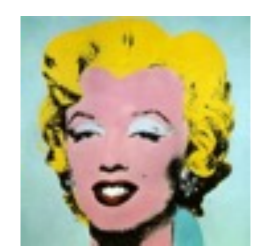


v_1

v_2

v_3



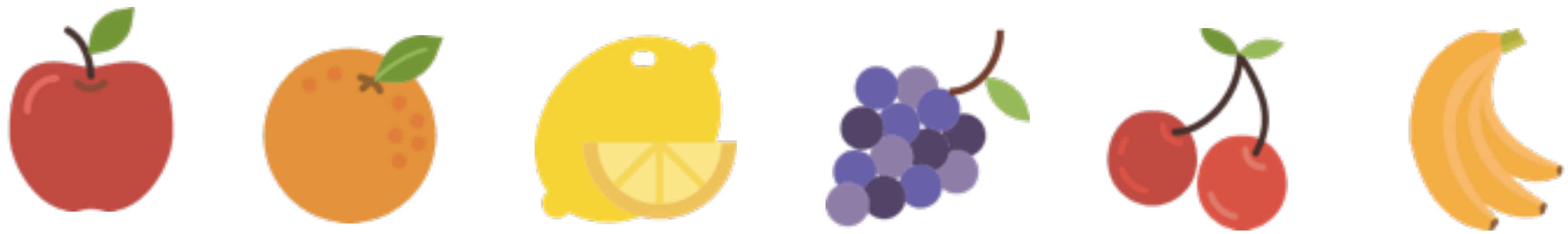
v_4

- Valuations $v_i : 2^N \to \mathbb{R}$

Market equilibrium

p_1 p_2 p_3 p_4 p_5 p_6

n goods



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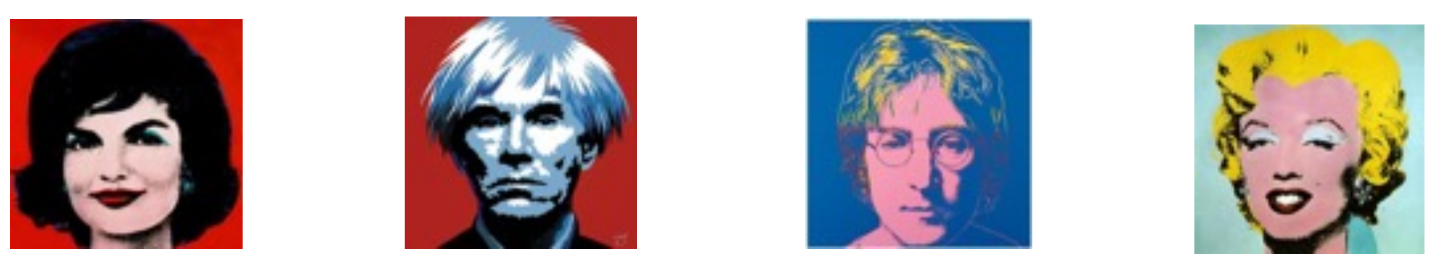

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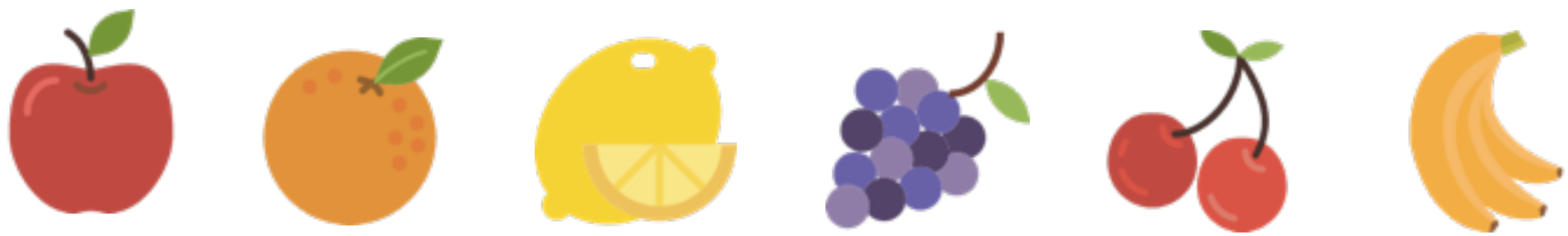

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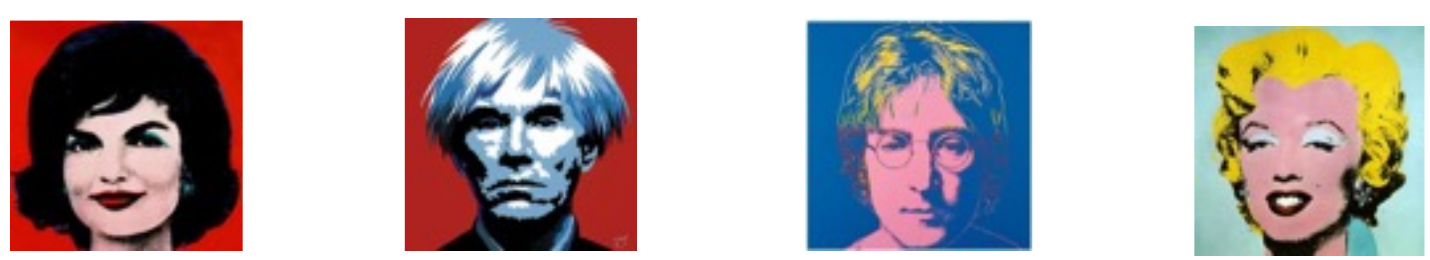

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- Valuations $v_i : 2^N \to \mathbb{R}$
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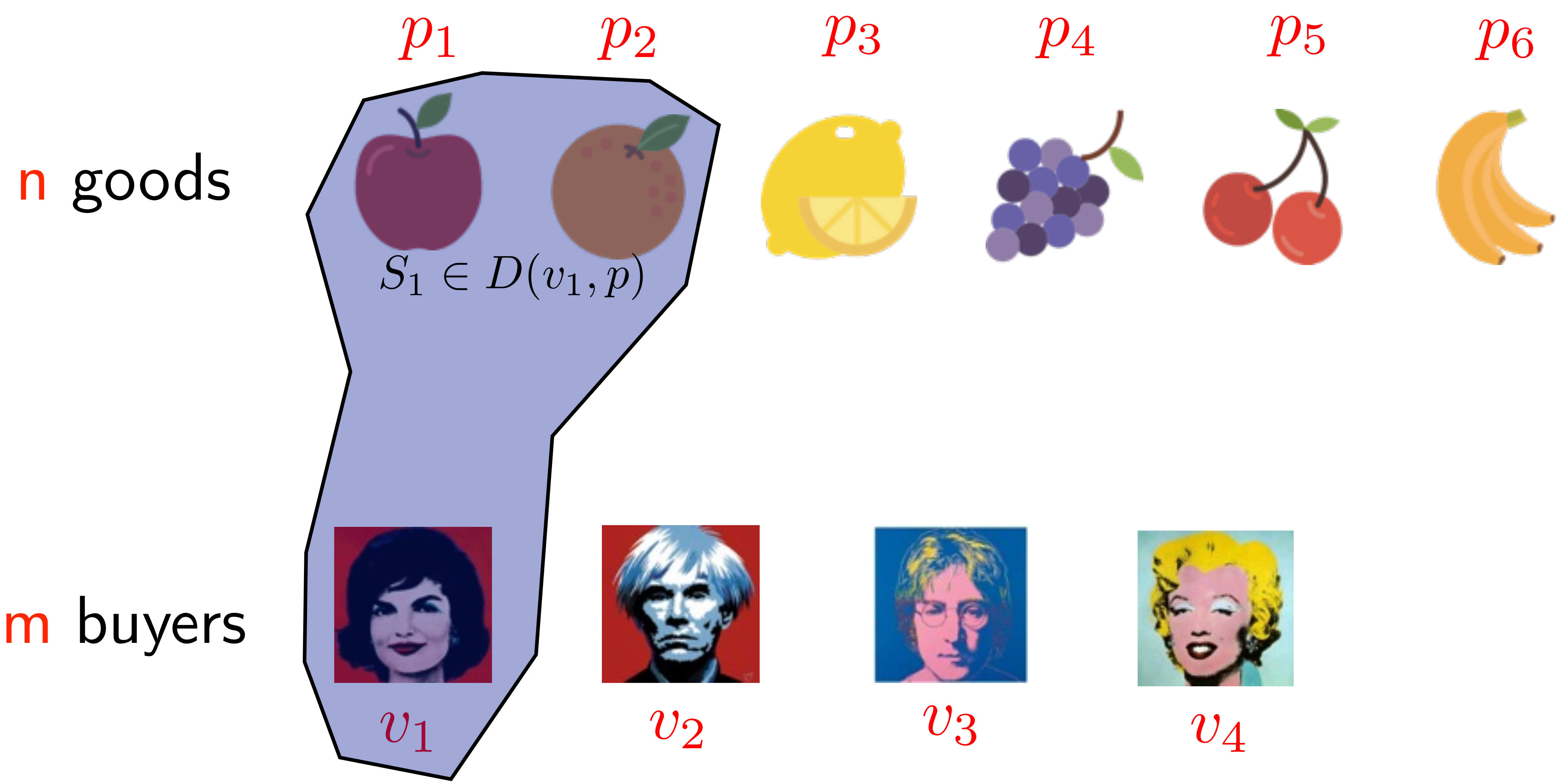
Market equilibrium



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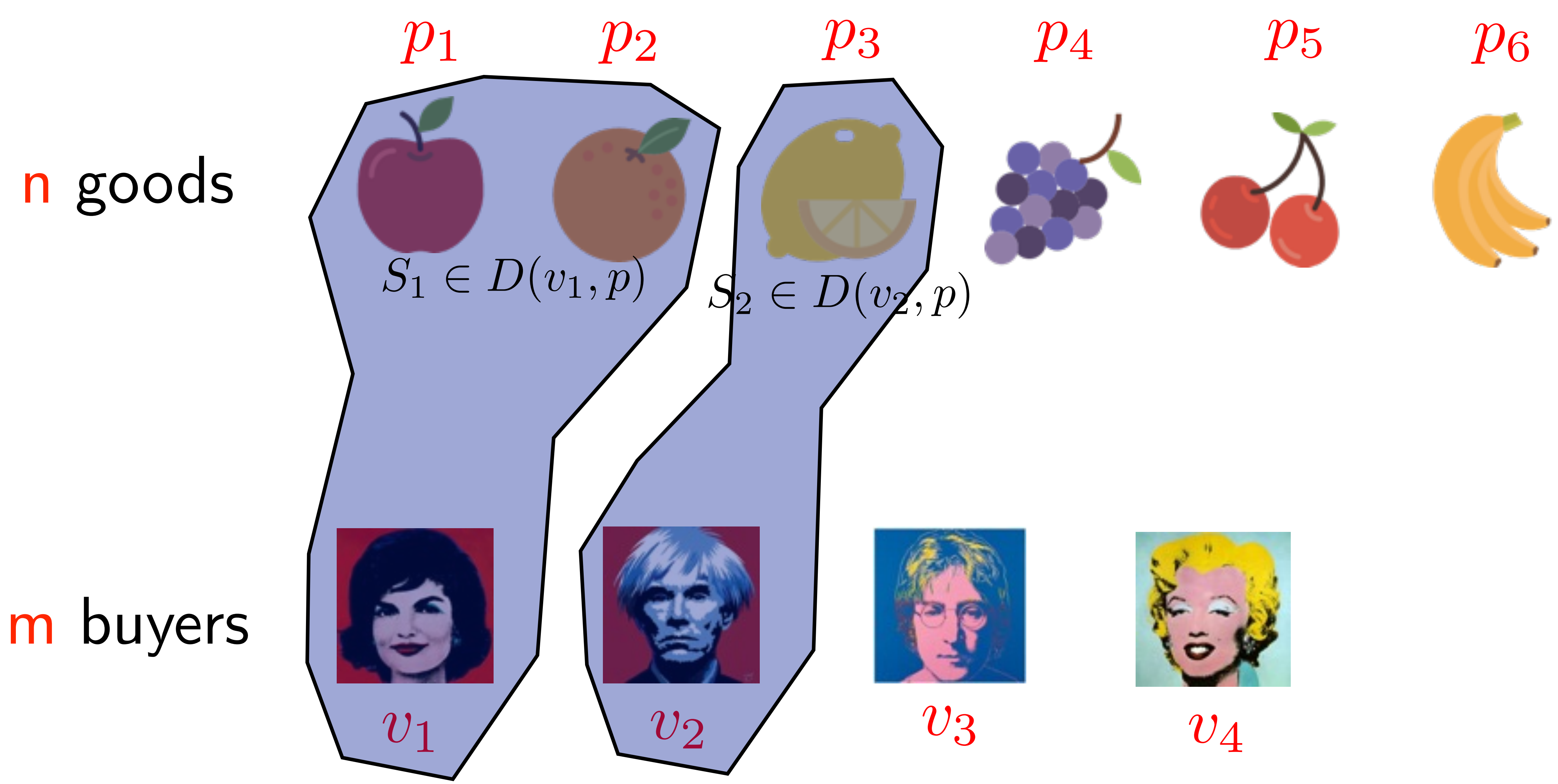

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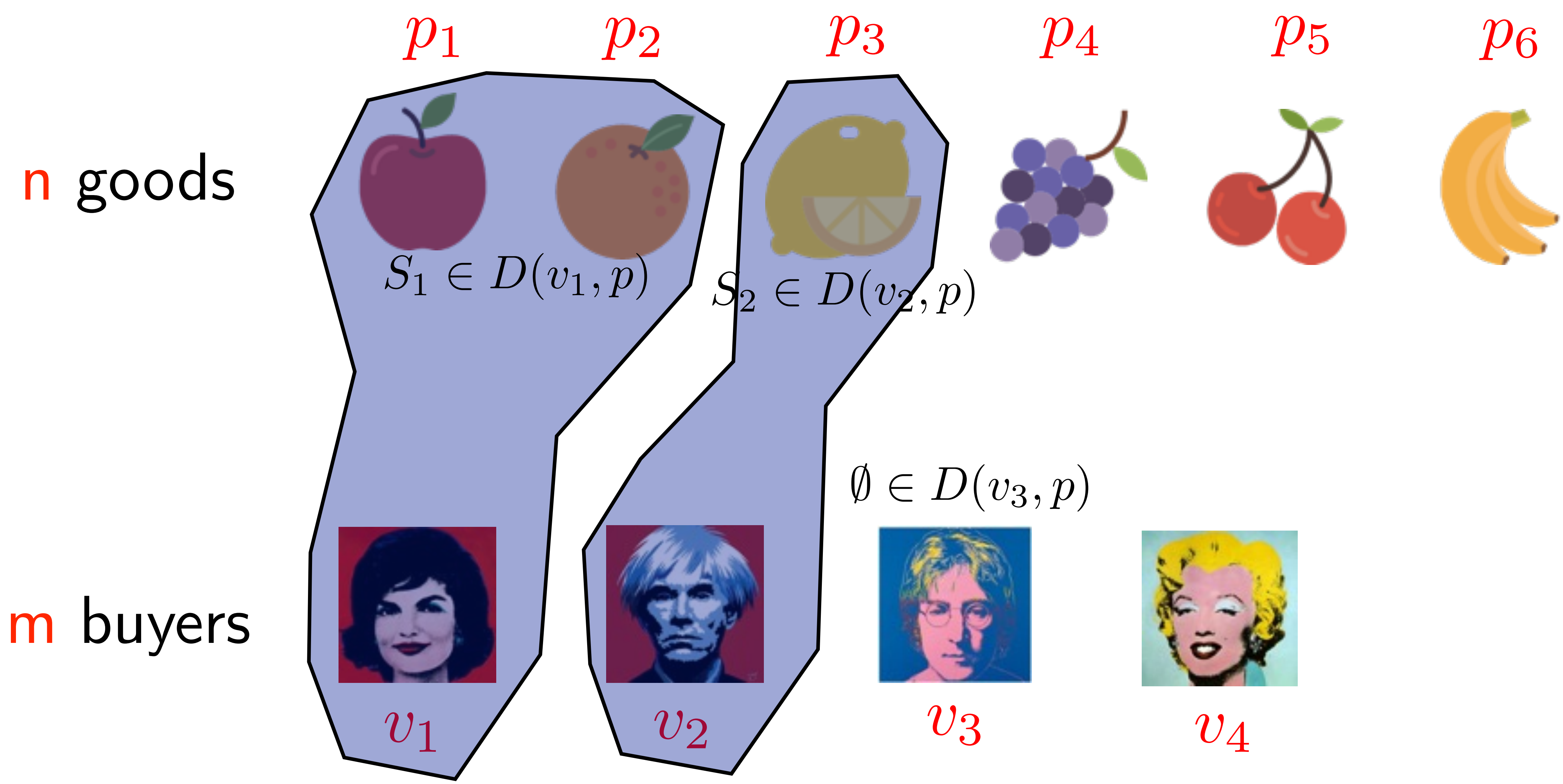

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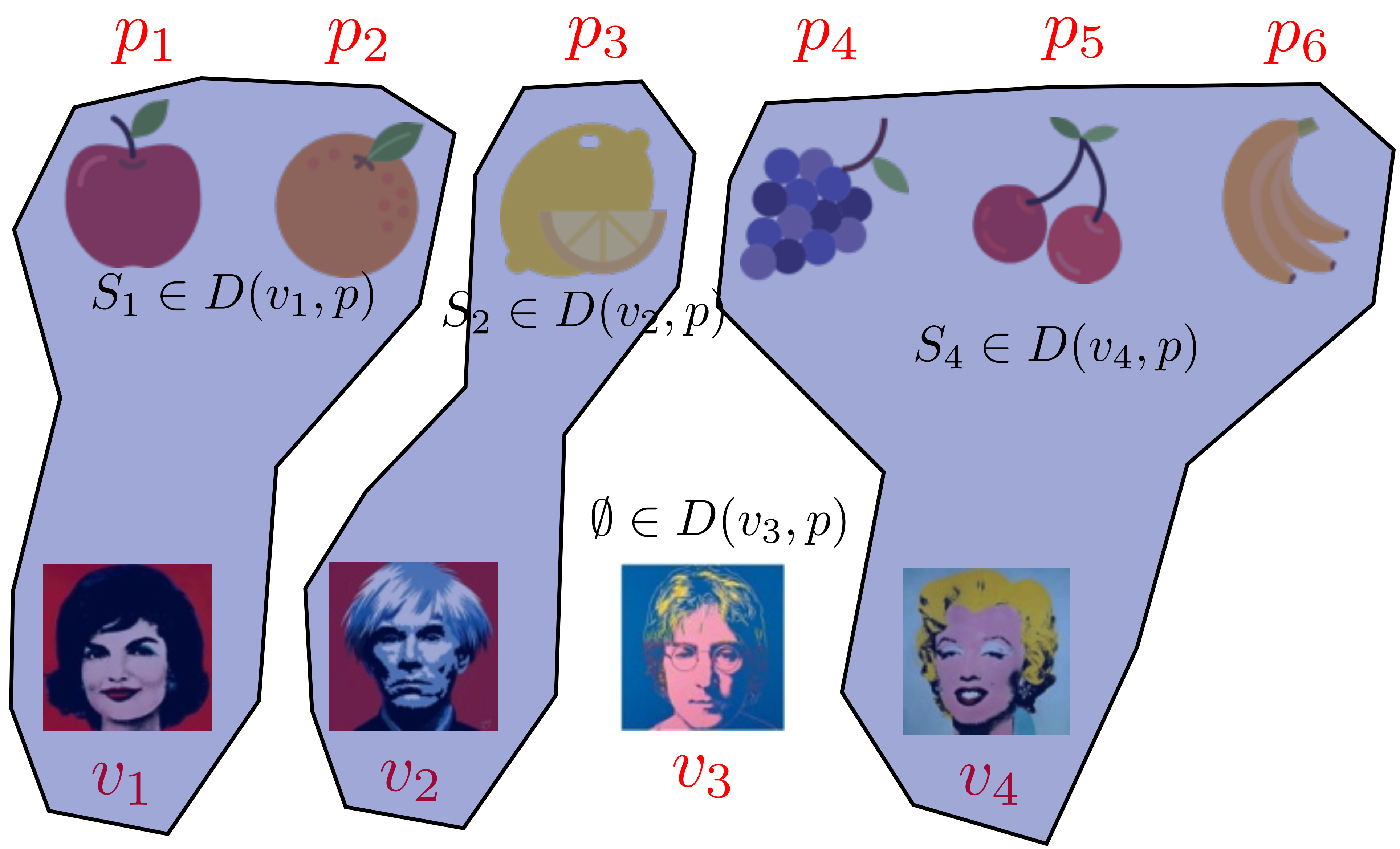

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Market equilibrium

- Market equilibrium: prices $p \in \mathbb{R}^n$ s.t. $S_i \in D(v_i, p)$ i.e. each good is demanded by **exactly** one buyer.

First Welfare Theorem: in equilibrium the welfare $\sum_i v_i(S_i)$ is maximized.

(proof: LP duality)

How do markets converge to equilibrium prices ?

How to compute a Walrasian equilibrium ?

How to access the input

Microscopic

Macroscopic

Telescopic

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Value oracle:

given i and S:

query $v_i(S)$.

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Macroscopic

Demand oracle:

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Telescopic

Aggregate Demand:

given p, query.

$\sum_i S_i; S_i \in D(v_i, p)$

Algorithms for computing equilibria (general case)

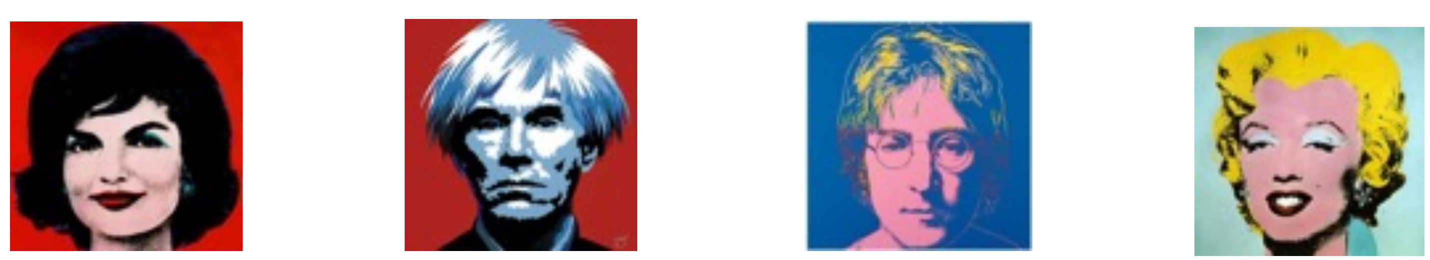
Algorithm	Oracle Access	Running time
tatonnement (trial-and-error) [Walras, Kelso-Crawford, ...]		

Walrasian tatonnement

p_1 p_2 p_3 p_4 p_5 p_6

n goods

m buyers



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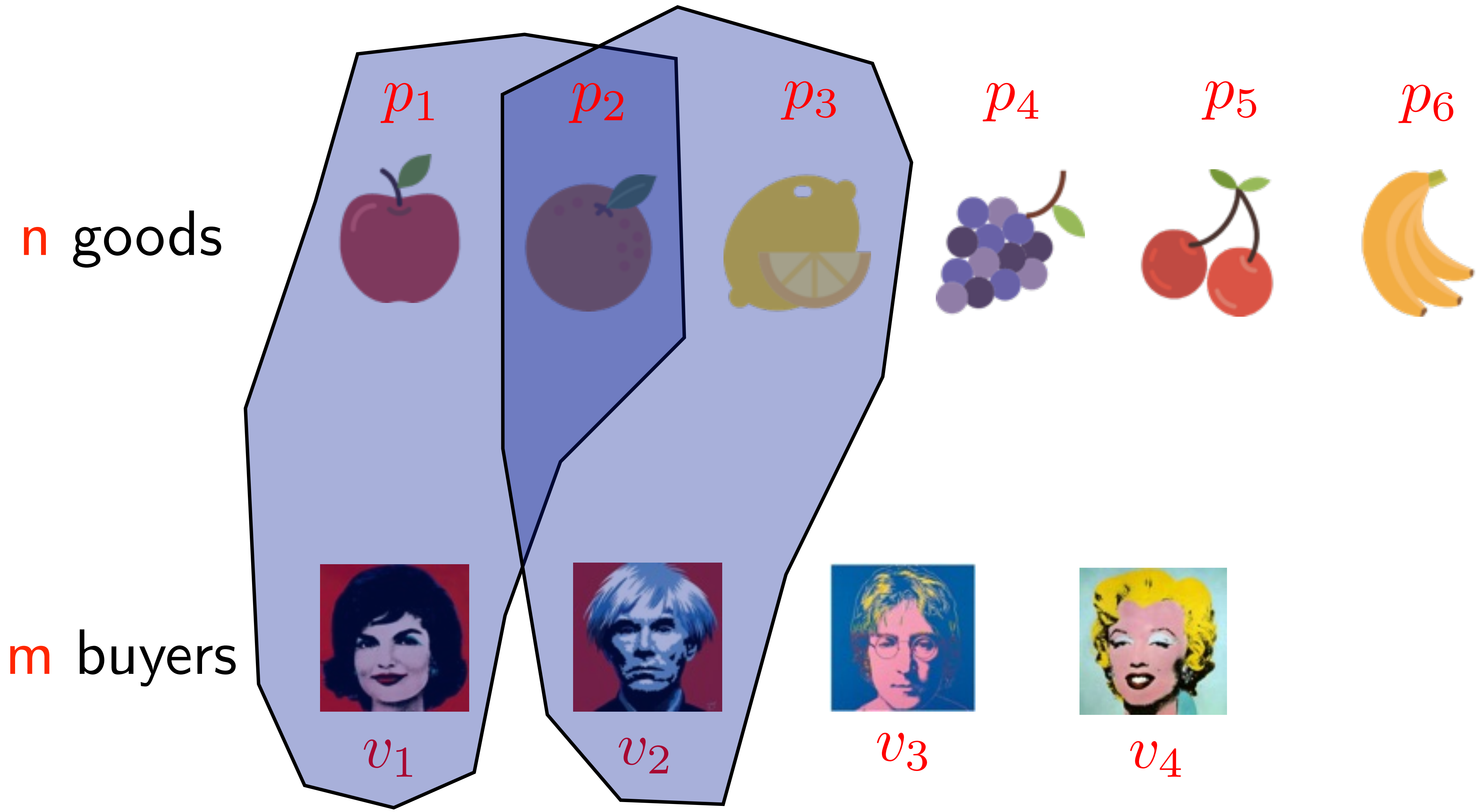

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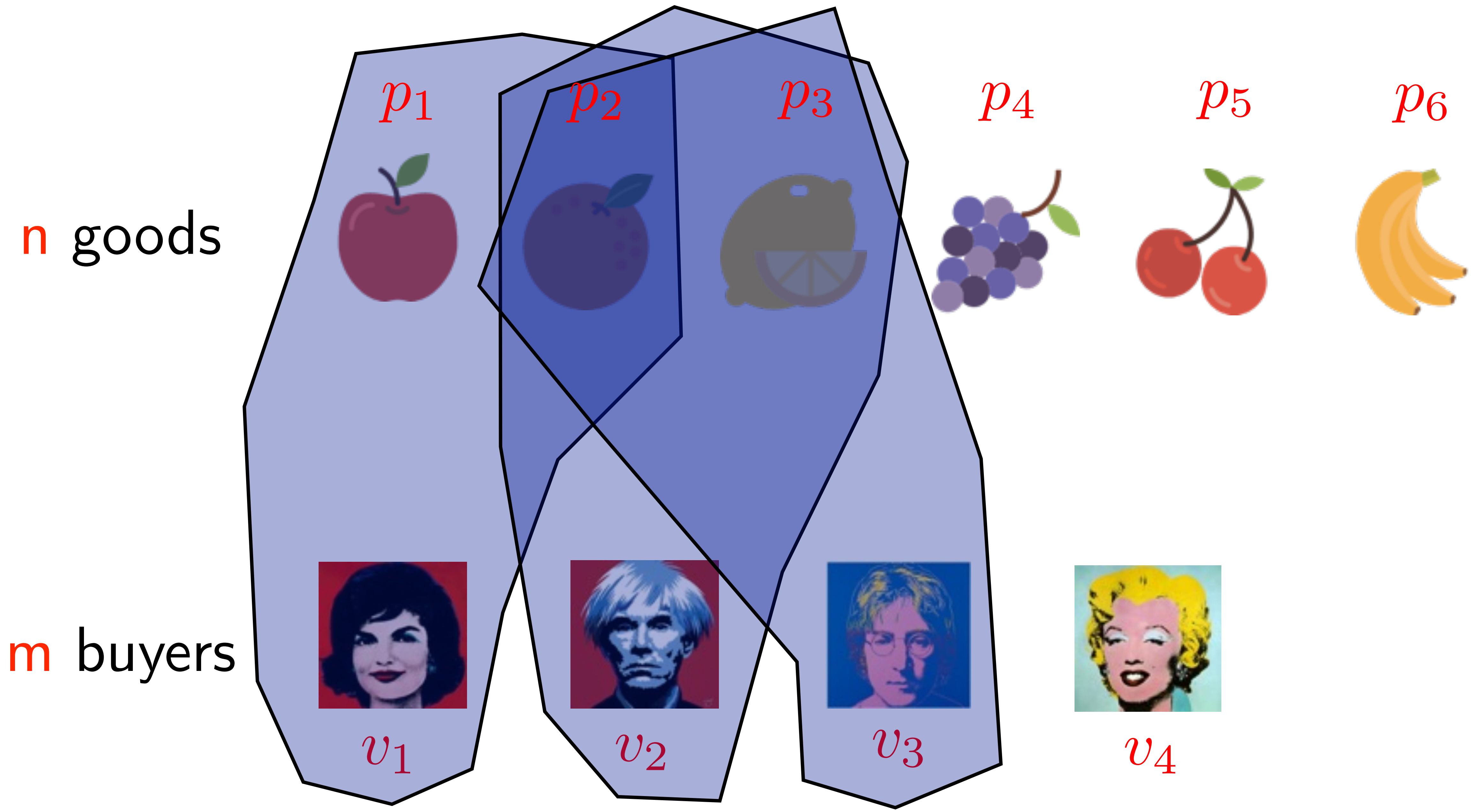

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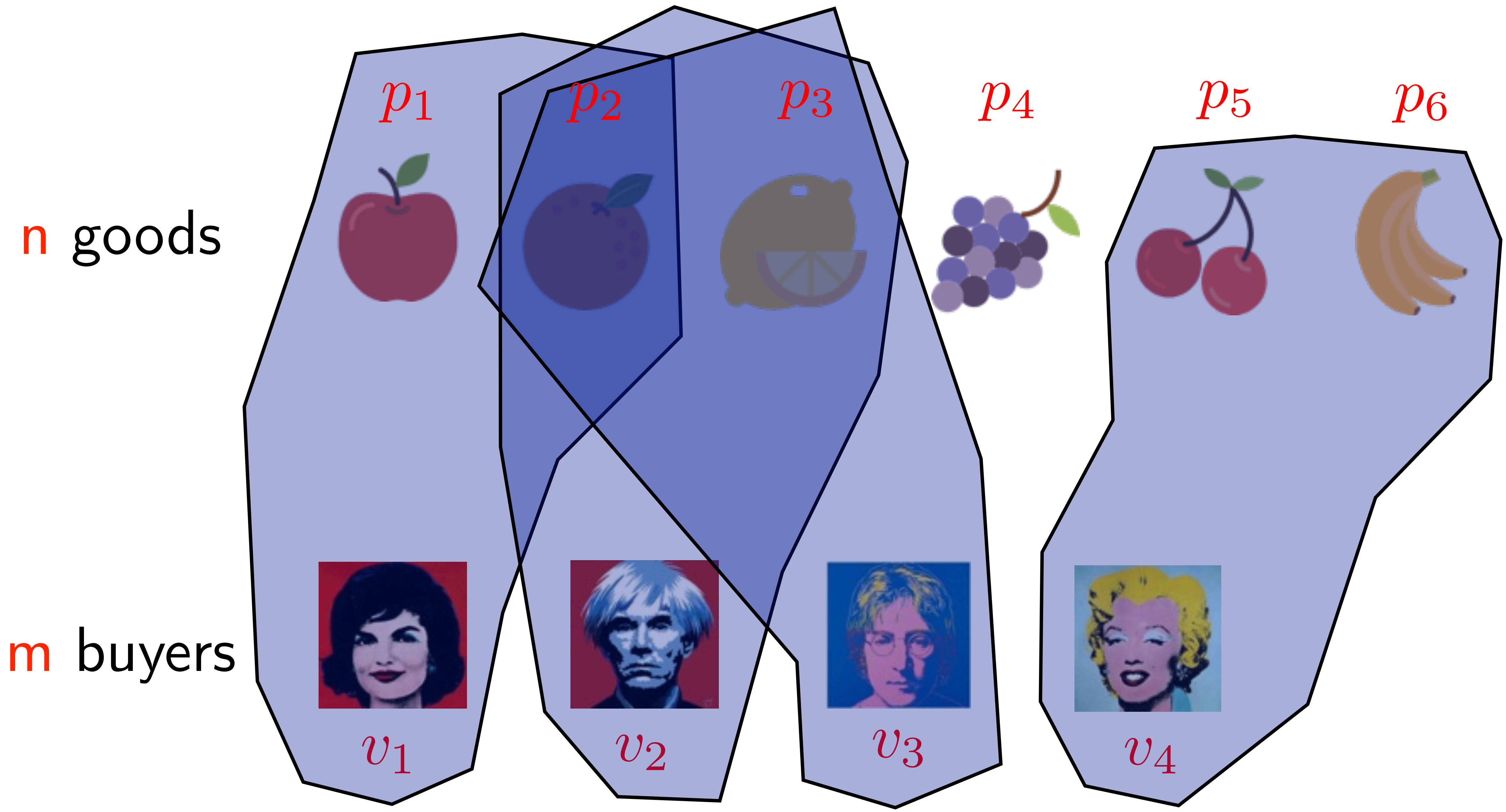

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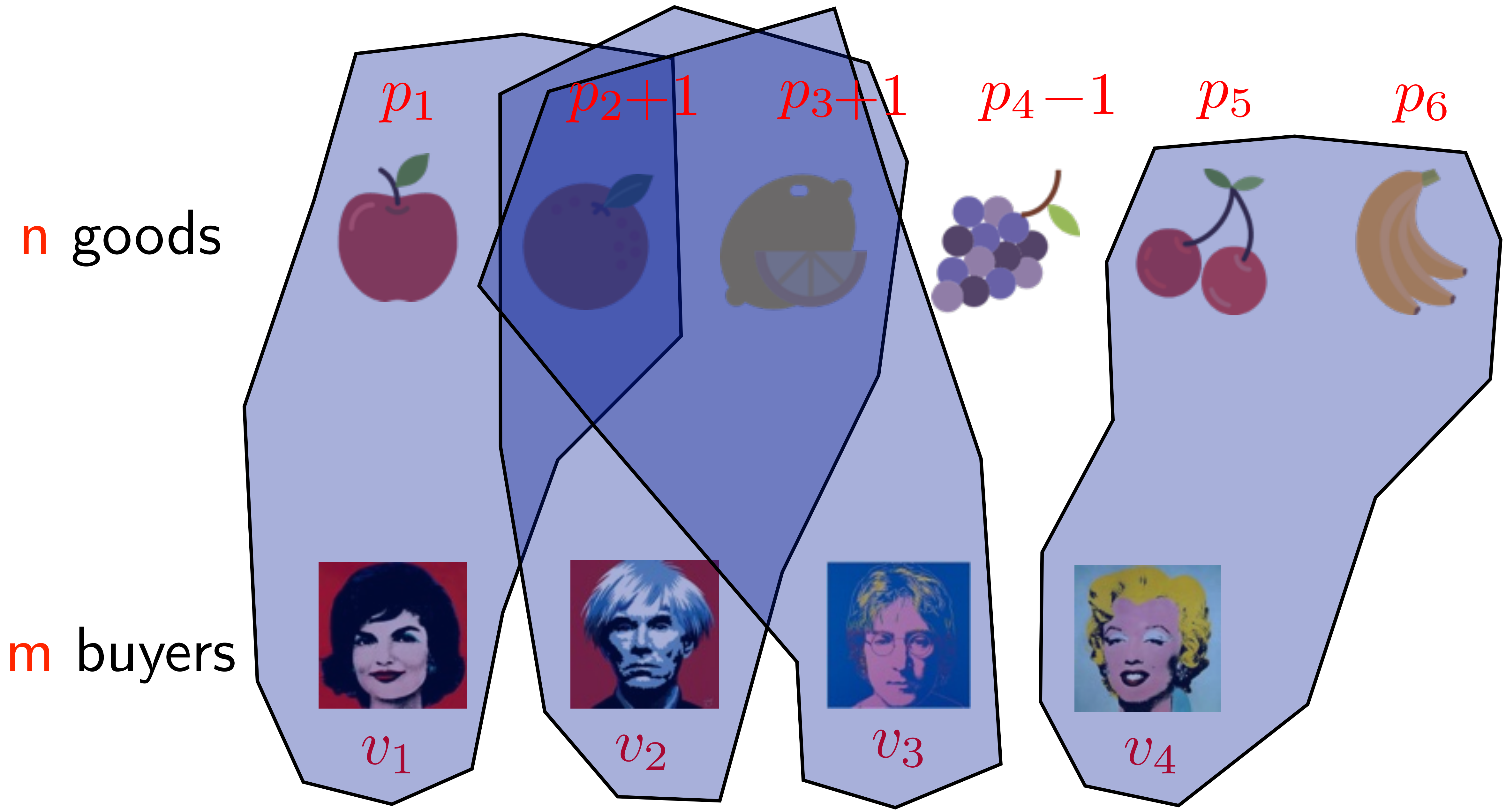
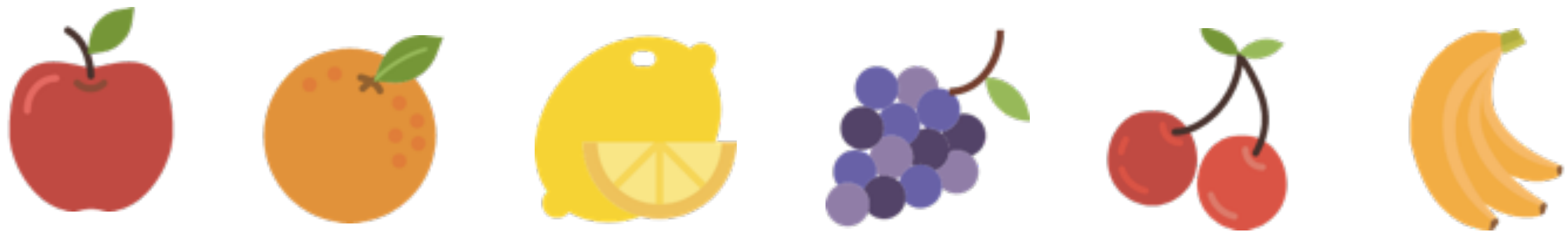


Walrasian tatonnement

n goods

p_1 $p_2{+}1$ $p_3{+}1$ p_4-1 p_5 p_6



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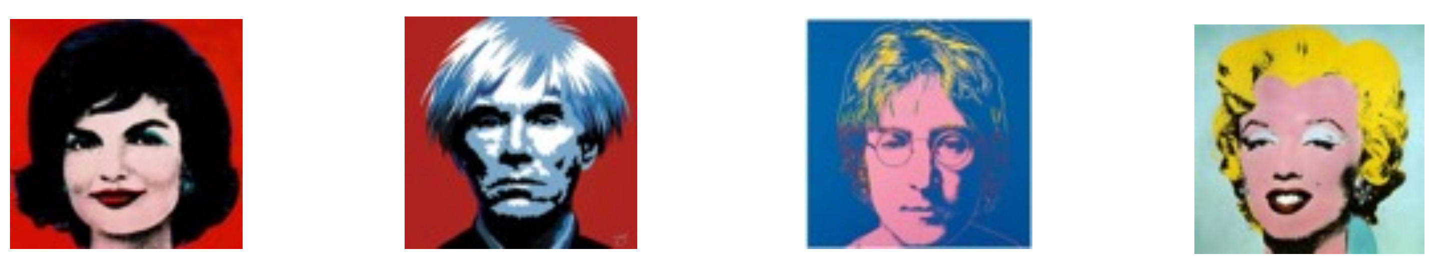

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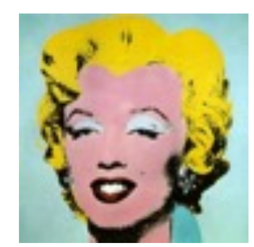

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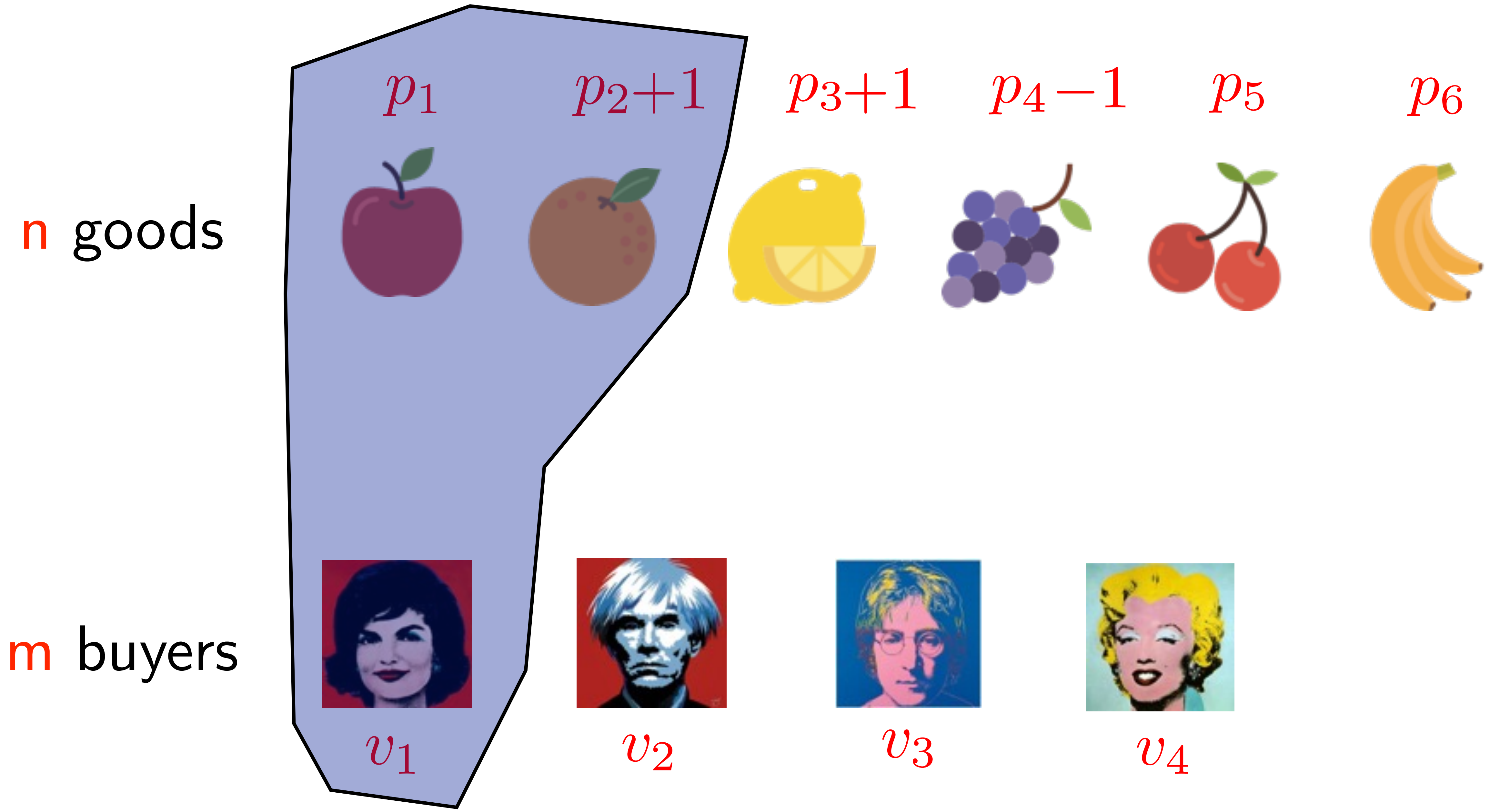

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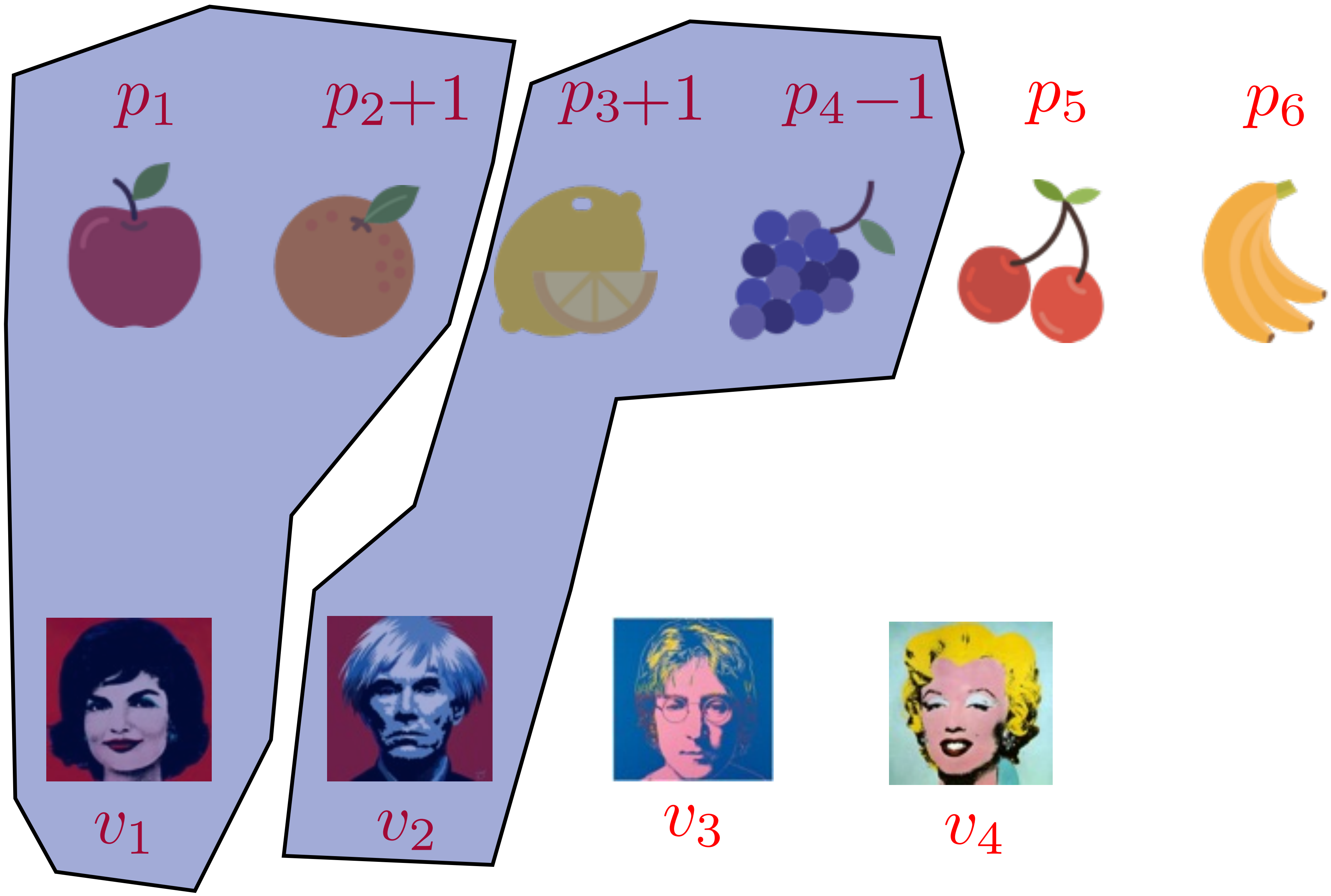

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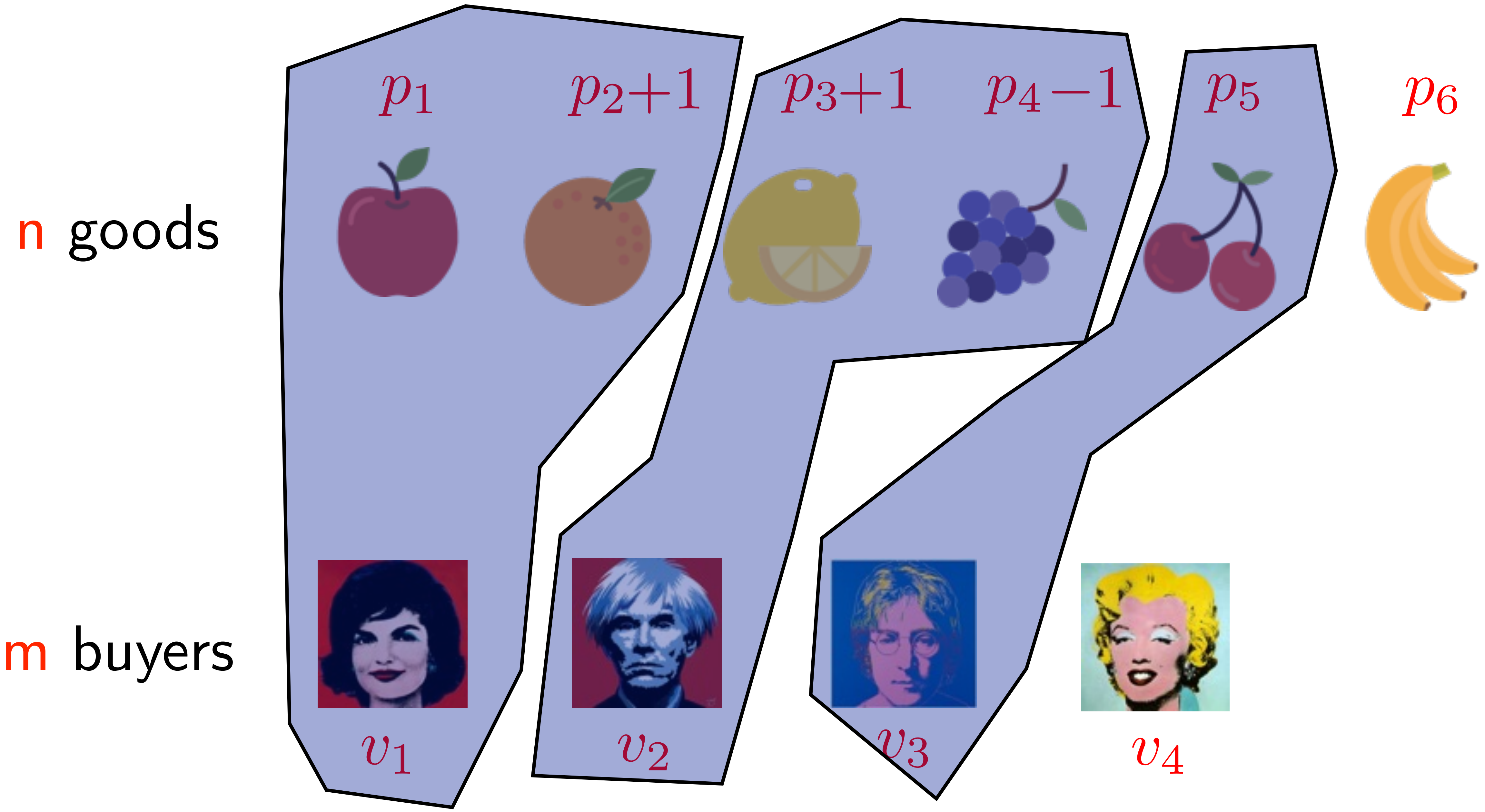

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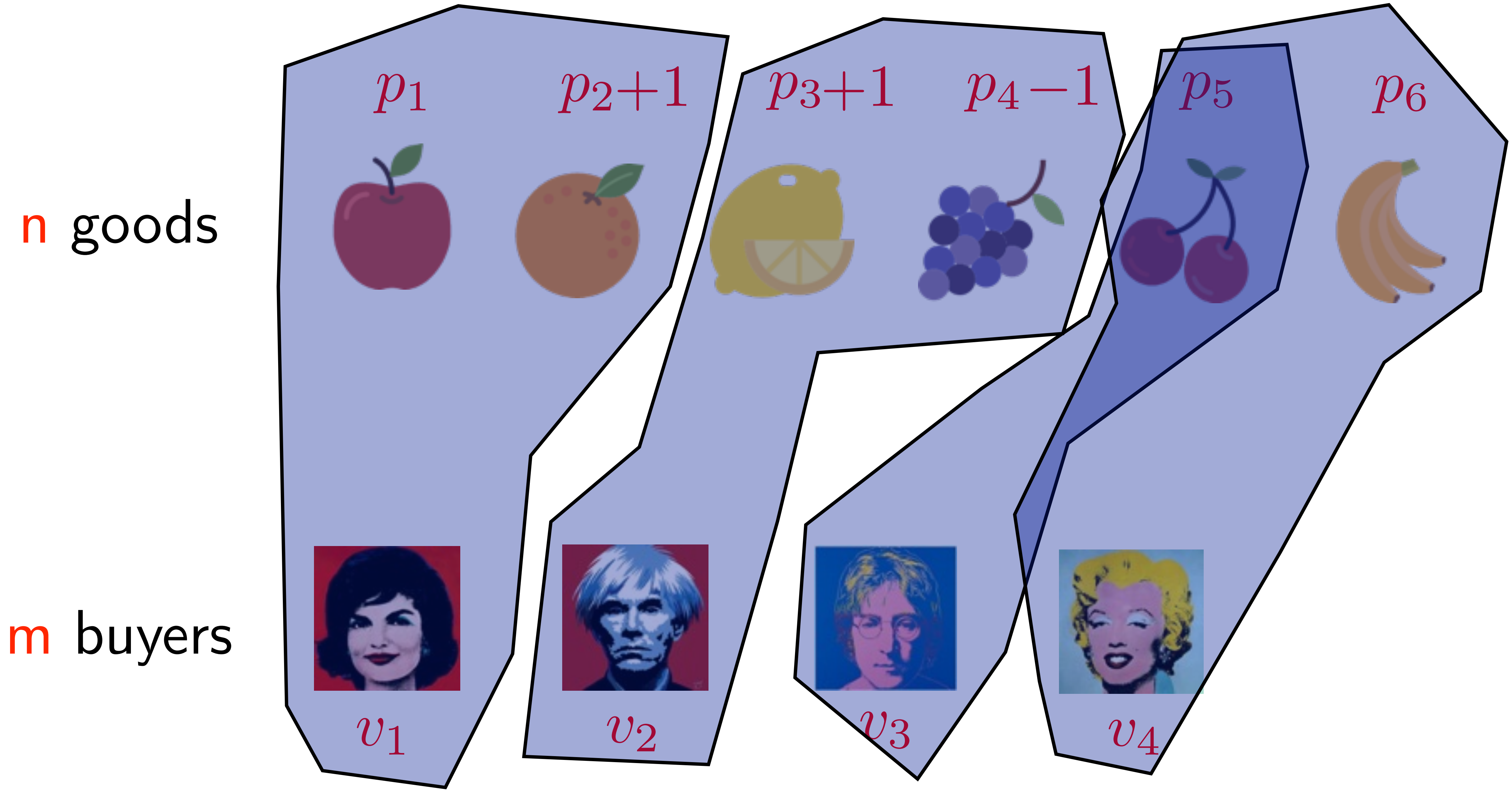

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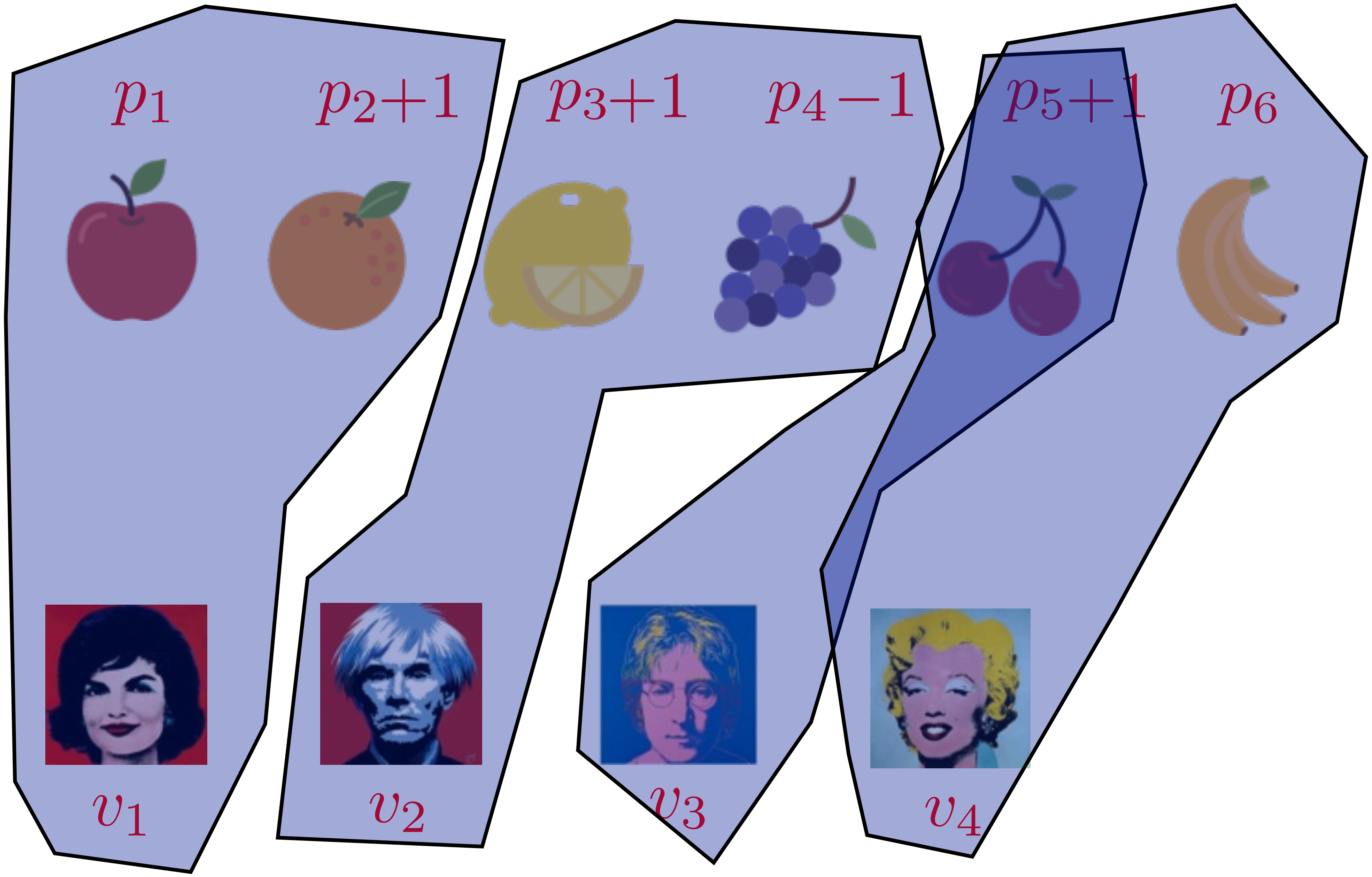

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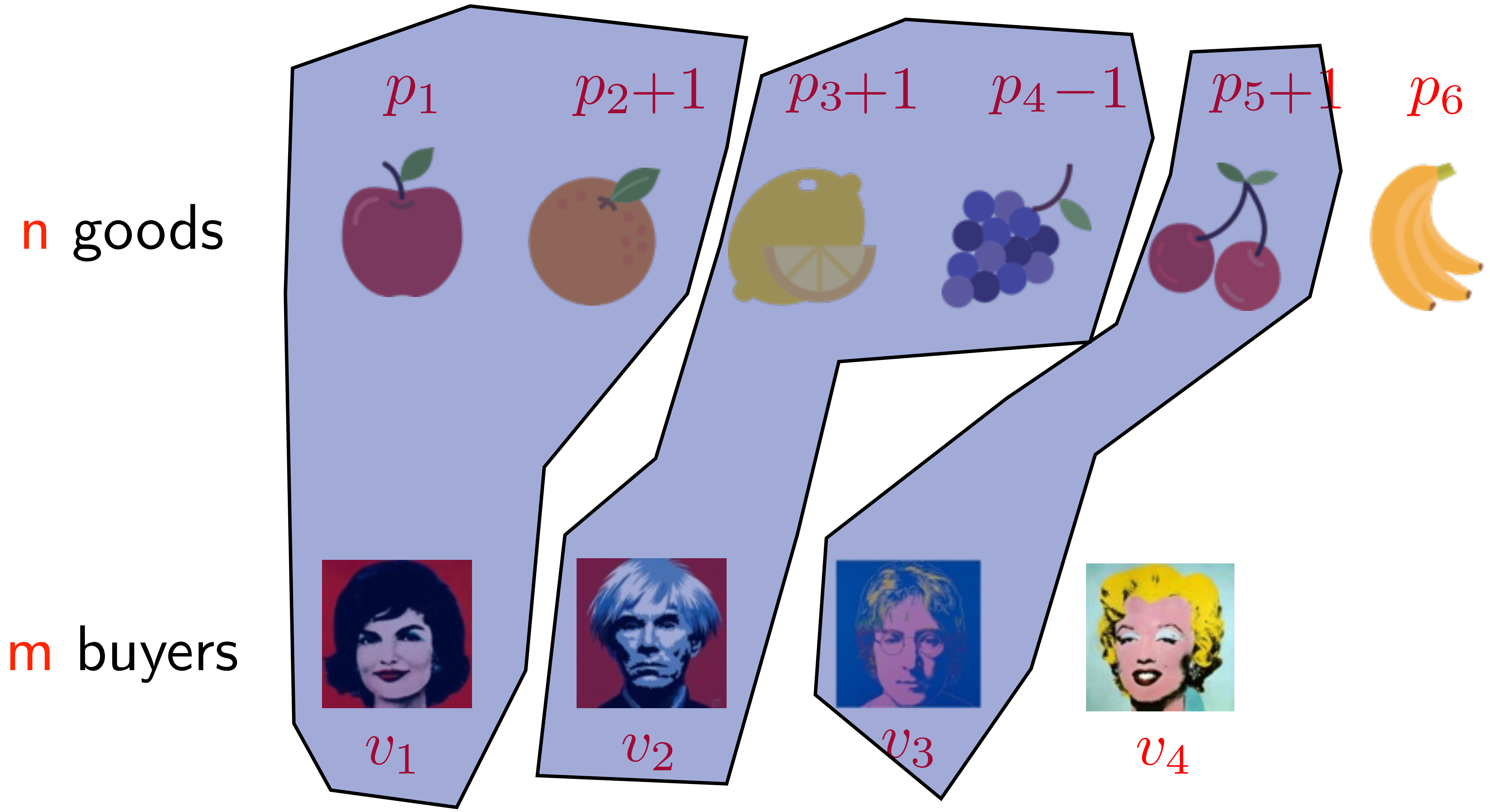

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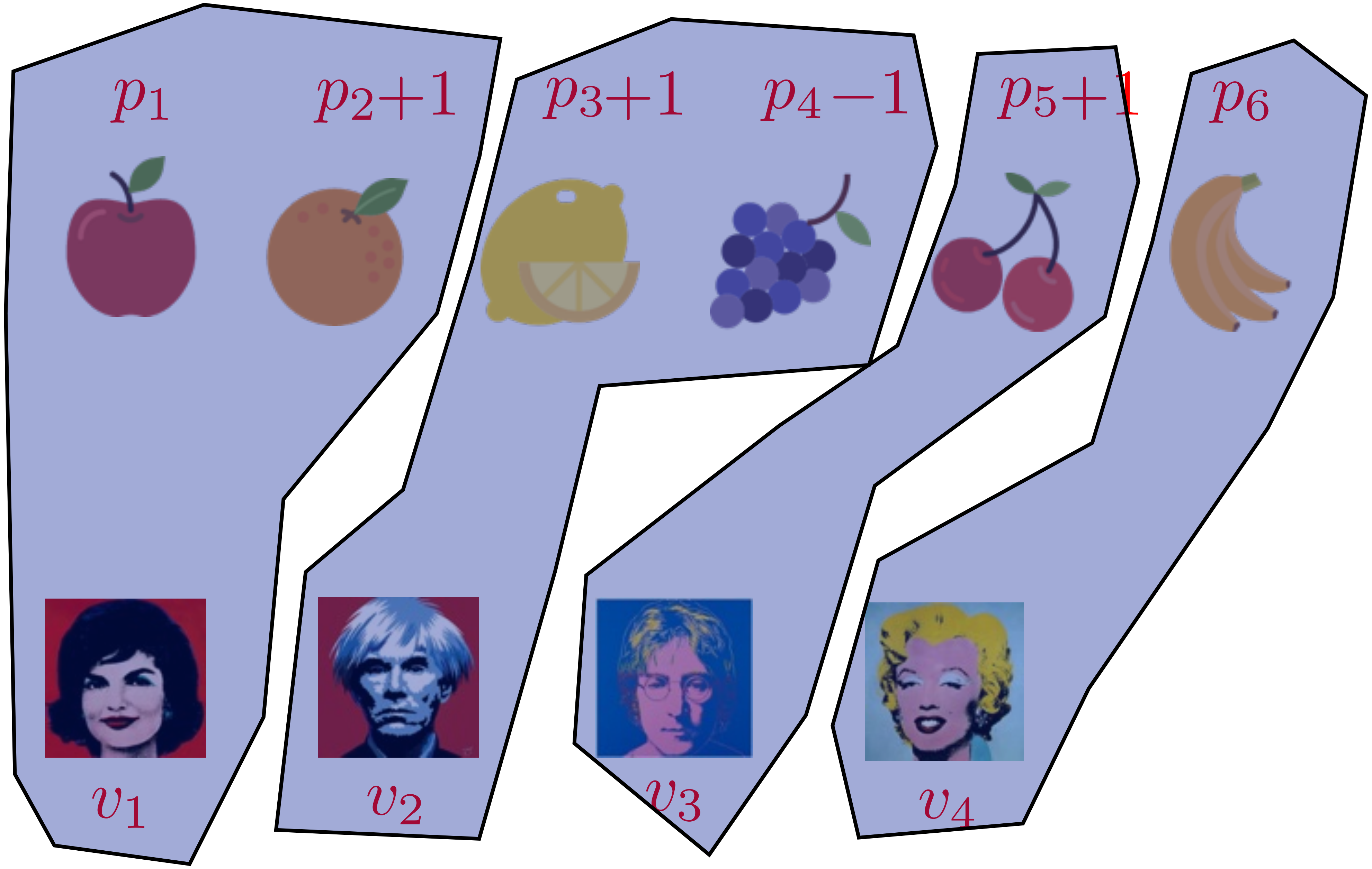

Gradient Descent Interpretation

- [Kelso-Crawford] analyzes it and shows convergence under a condition called gross substitutes.
  - pseudo poly algorithm

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- [Ausubel] defined the potential:

$$f(p) = \sum_i \max_S [v_i(S) - p(S)] + p([n])$$

  such that gradient descent is exactly tatonnement:

$$\partial_j f(p) = 1 - [\text{total demand for } j]$$

- If equilibrium exists then equil prices = $\operatorname{argmin} f(p)$

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this paper	aggregate demand	poly time $\tilde{O}(n^2 \cdot T_{AD} + n^5)$

From LP to convex optimization

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$$\min \sum_i u_i + p([n])$$

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- Two issues with black box application:
  - **Evaluate f**: ellipsoid and cutting plane need $f(p), \partial f(p)$
  - **Approximation**: give only approximate solutions

From LP to convex optimization

- **Optimizing only using the gradient**
  We adapt the cutting plane algorithm of Lee-Sidford-Wong'15 to optimize f using only $\partial f(p)$

- **Obtaining exact solutions**
  - Exact solution is only known for LPs [Khachiyan]
  - idea: explore the connection of this program and LP
  - But we have restricted access to constraints (only via aggregate demand oracle)
  - Only a restricted perturbation is enough.

Gross substitutes case

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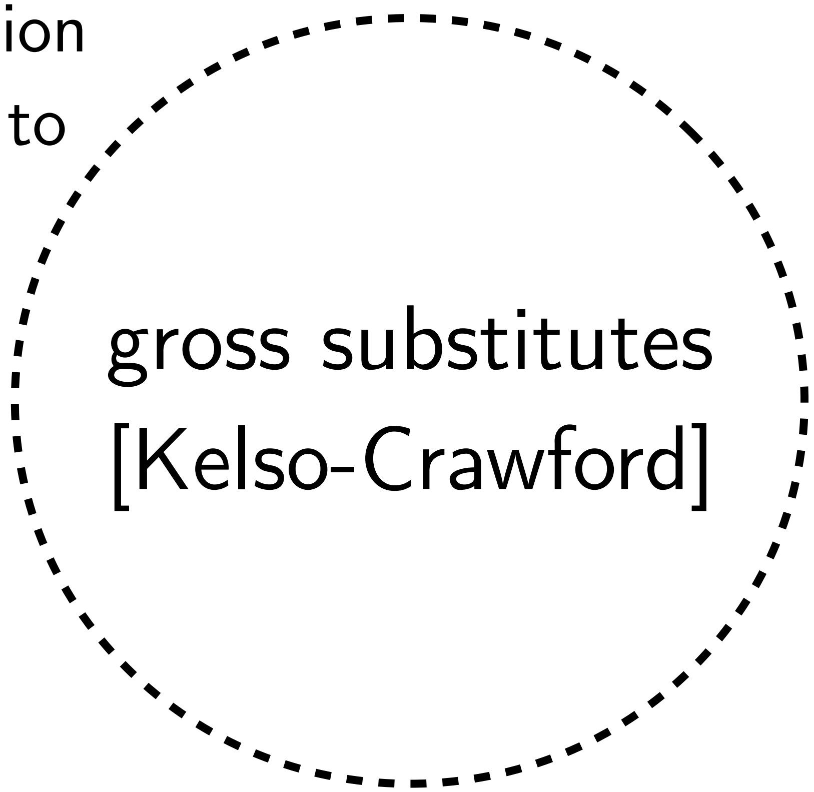
necessary and "sufficient" condition for tatonnement to converge

"increase in the price for one good doesn't decrease demand for other good."



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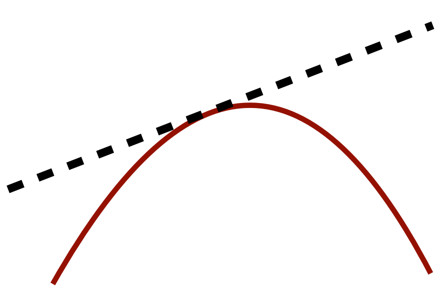
discrete concavity [Murota-Shioura]



local certificate of global optimality

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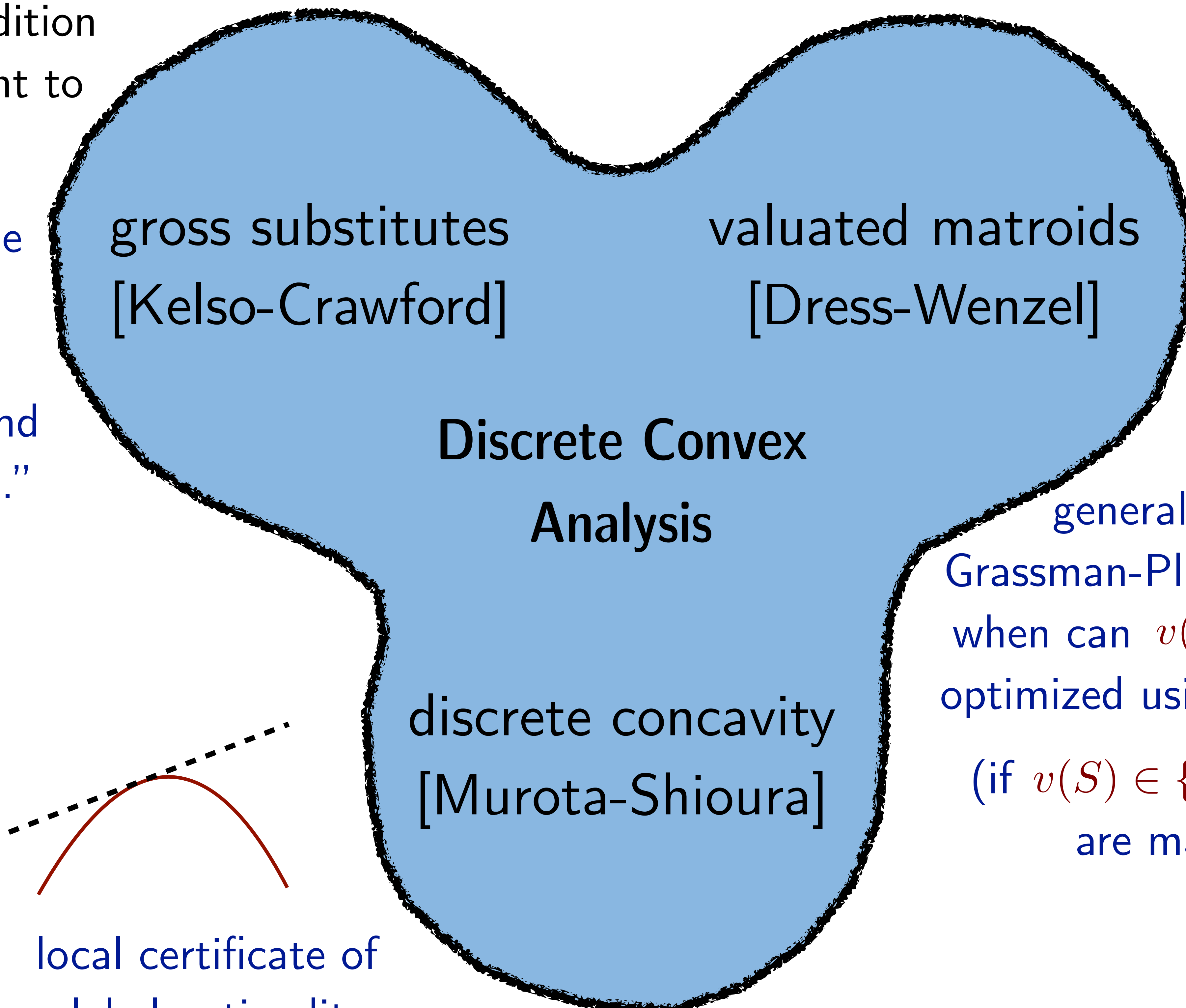

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this paper	value oracle	$\tilde{O}((mn + n^3) \cdot T_V)$

Improving the algorithm for gross substitutes

- Better rounding using structure of gross substitutes gets us to $\tilde{O}(n \cdot T_{AD} + n^3)$
  - plugging $T_{AD} = O(mn^2 \cdot T_V)$ we get $\tilde{O}(mn^3 \cdot T_V)$

- Regularization: gradients are expensive to compute.
  - it takes $O(n^2 \cdot T_V)$ to run Greedy for each buyer.
  - gradients are cheap near the optimal
  - re-use computation from one step to the next
  - we only need precise gradients near the optimum

$$\hat{f}(p) = \sum_i [\max_S v_i(S) - p(S) + \epsilon|S|] + p([n]) - \epsilon n$$

Improving the algorithm for gross substitutes

- Regularized objective:

$$\hat{f}(p) = \sum_i [\max_S v_i(S) - p(S) + \epsilon|S|] + p([n]) - \epsilon n$$

- Same optimal value
- Very accurate near the optimal value, directionally correct for other values.
- Takes only $O(n^2)$ time to compute with $O(mn)$ pre-processing.

Conclusion

- Market equilibrium can be computed:
  - only very aggregated information
  - in $\tilde{O}(n)$ calls to this oracle.

- Questions to think about:
  - Markets that change over time ? New items, new buyers, ... How to update market equilibrium.
  - Strongly poly time algorithms.