Gross Substitutes Tutorial

Part II: Economic Implications + Pushing the Boundaries

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INBAL TALGAM-COHEN → TECHNION CS

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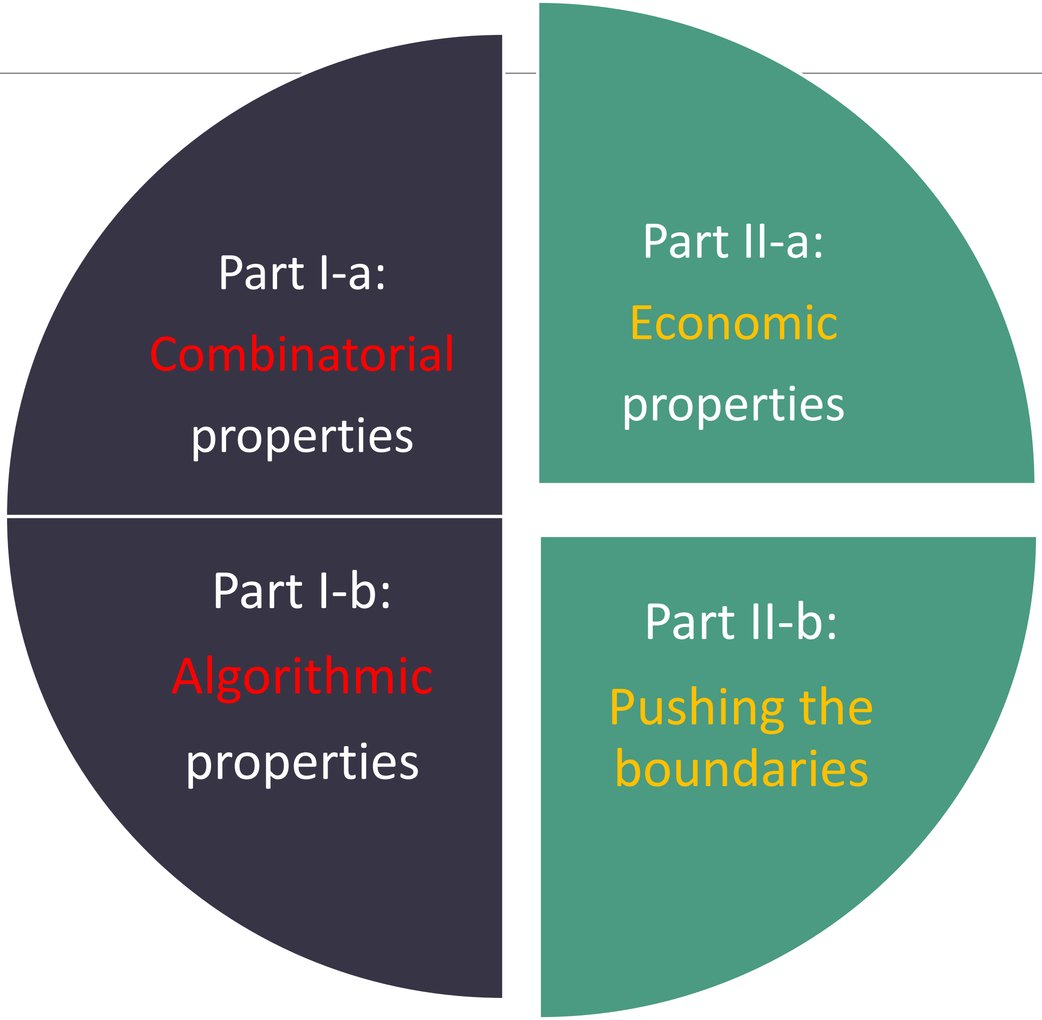
Roadmap


Part I-a:
Combinatorial
properties
Part I-b:
Algorithmic
properties
Part II-a:
Economic
properties
Part II-b:
Pushing the
boundaries

Previously, in Part I

Remarkable combinatorial + algorithmic properties of GS

1 GS valuation:

- Combinatorial exchange properties
- Optimality of greedy & local search algorithms for DEMAND

n GS valuations (= market):

- Walrasian market equilibrium existence
- WELFARE-MAX (and pricing) computationally tractable

Plan for Part II

1. Economic implications: Central results in market design that depend on the nice properties of GS

2. Pushing the boundaries of GS:
   - Robustness of the algorithmic properties
   - Extending the economic properties (networks and beyond)

Classic theory (and some recent insights)

State-of-the-art and open challenges

<u>Disclaimer</u>:

- Literature too big to survey comprehensively

Motivation

GS assumption fundamental to market design with indivisible items

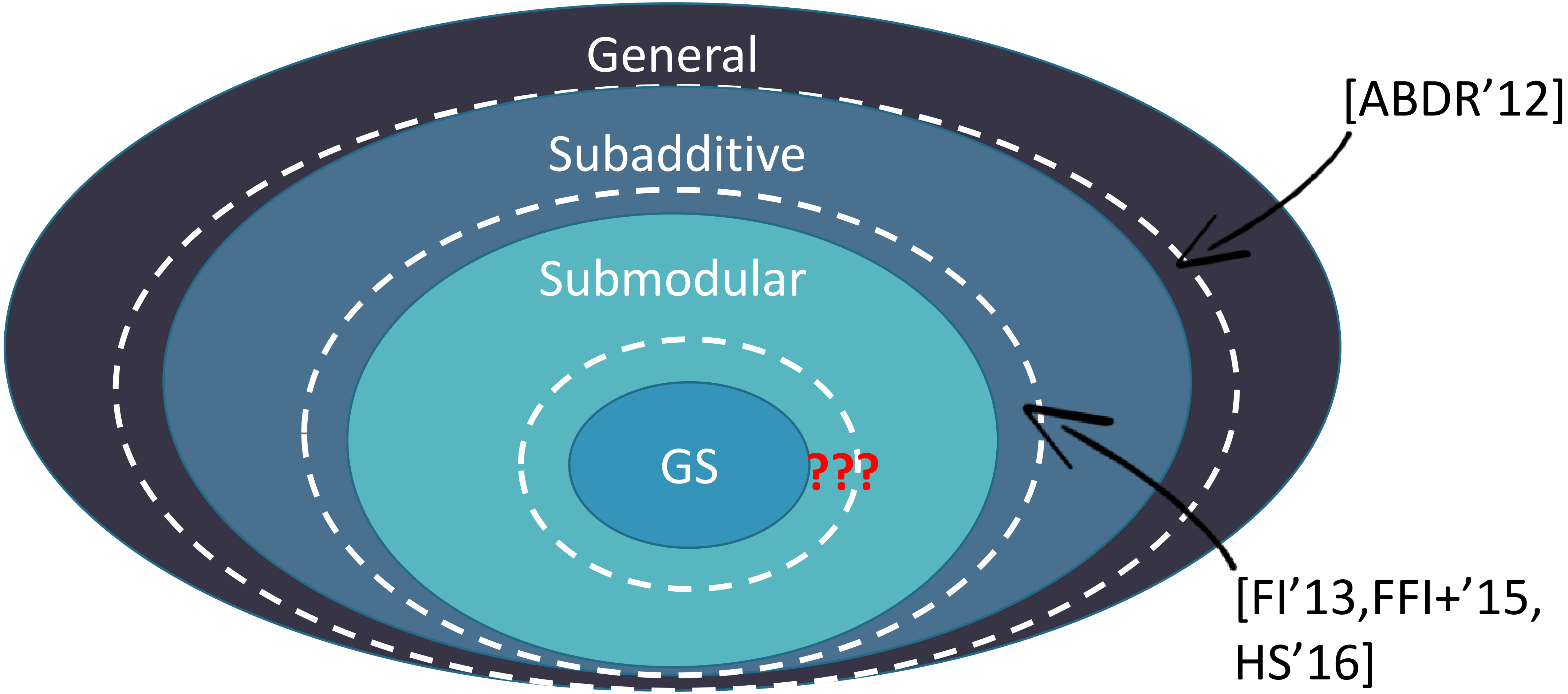
- Sufficient (and in some sense necessary) for the following results:

1. Equilibrium prices exist and have a nice lattice structure
2. VCG outcome is revenue-monotone, stable (in the core)
3. "Invisible hand" – prices coordinate "typical" markets

- (GS preserved under economically important transformations)

- Interesting connection between economic, algorithmic properties

More Motivation: Uncharted Territory



Recall Our Market Model

m buyers M (notation follows [Paes Leme'17])

$m+1$ players in the grand coalition $G = M \cup \{0\}$

- player $i = 0$ is the seller

n indivisible items N

Allocation $\mathcal{S} = (S_1 \ldots, S_m)$ is a partition of items to m bundles

Prices: $p \in \mathbb{R}^n$ is a vector of item prices; let $p(S) = \sum_{j \in S} p_j$

- So $p(N)$ = seller's utility (revenue) from clearing the market

Recall Our Buyer Model

Buyer i has valuation $v_i: 2^N \to \mathbb{R}$

Fix item prices p

- If buyer i gets S_i, her quasi-linear utility is

$$\pi_i = \pi_i(S_i, p) = v_i(S_i) - p(S_i)$$

- S_i is in buyer i's demand given p if

$$S_i \in \arg\max_S \pi_i(S, p)$$

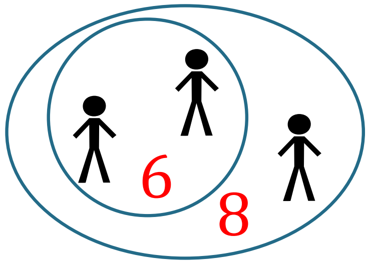
Preliminaries

1. THE CORE
2. SUBMODULARITY ON LATTICES
3. FENCHEL DUAL

Preliminaries: The Core

Consider the cooperative game (G, w):

- players G
- coalitional value function $w: 2^G \to \mathbb{R}$



$\pi = (2, 3, 3)$

π = utility profile associated with an outcome of the game

Coalition $C \subseteq G$ will not cooperate ("block") if $\sum_{i \in C} \pi_i < w(C)$

Definition: π is in the core if no coalition is blocking, i.e.,

$$\sum_{i \in C} \pi_i \geq w(C) \text{ for every } C$$

Preliminaries: Lattices

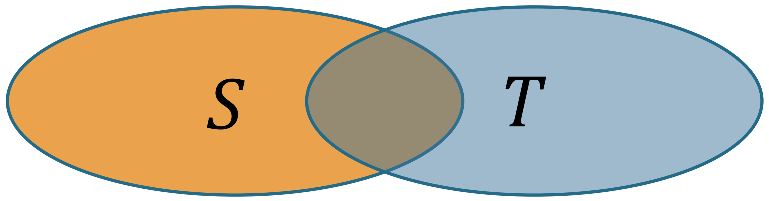
Lattice = partially ordered elements $(X, \preccurlyeq)$ with "join"s, "meet"s $\in X$

- Join $\vee$ of 2 elements = smallest element that is $\succcurlyeq$ both
- Meet $\wedge$ of 2 elements = largest element that is $\preccurlyeq$ both

Preliminaries: Lattices

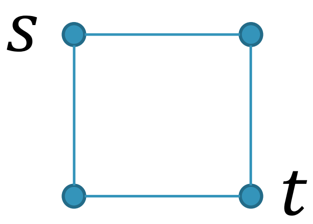
$(2^N, \subseteq)$ is a lattice:

- Join of $S, T \in 2^N$ is $S \cup T$
- Meet of $S, T \in 2^N$ is $S \cap T$



$(\mathbb{R}^n, \leq)$ is a lattice:

- Join of $s, t \in \mathbb{R}^n$ is their component-wise $\max$
- Meet of $s, t \in \mathbb{R}^n$ is their component-wise $\min$



Can naturally define a product lattice

- E.g. over $2^N \times \mathbb{R}^n$, or $\mathbb{R}^n \times 2^M$ = prices x coalitions

Preliminaries: Submodularity on Lattices

Definition:

f is submodular on a lattice if for every 2 elements s, t,

$$f(s) + f(t) \geq f(s \vee t) + f(s \wedge t)$$

Preliminaries: Fenchel Dual

$v: 2^N \to \mathbb{R}$ = valuation

<u>Definition</u>: The Fenchel dual $u: \mathbb{R}^N \to \mathbb{R}$ of v maps prices to the buyer's max. utility under these prices

$$u(p) = \max_S \{v(S) - p(S)\} = \max_S \{\pi(S, p)\}$$

<u>Theorem</u> [Ausubel-Milgrom'02]: v is GS iff its Fenchel dual is submodular

Preliminaries: Fenchel Dual & Config. LP

$$\max_{x}\{\sum_{i,S} x_{i,S} v_i(S)\}$$

$$\text{s.t.} \sum_{S} x_{i,S} \le 1 \; \forall i$$

$$\sum_{i,S:j\in S} x_{i,S} \le 1 \forall j$$

$$x \ge 0$$

Using Fenchel dual $u_i(\cdot)$:

$$\min_{p}\{\sum_{i} u_i(p) + p(N)\}$$

Maximize welfare (sum of values)

s.t. feasibility of allocation

Minimize total utility (including seller's)

s.t. buyers maximizing their utility

Preliminaries: Fenchel Dual

From previous slide: For GS, the maximum welfare is equal to

$$\min_p \{\sum_{i \in M} u_i(p) + p(N)\}$$

where $u_i(\cdot)$ = Fenchel dual

Applying to buyer i and bundle S we get the duality between v_i, u_i:

$$v_i(S) = \min_p \{u_i(p) + p(S)\}$$

1. Economic Implications of GS

Economic Implications of GS

1. Equilibrium prices form a lattice
2. VCG outcome monotone, in the core
3. Prices coordinate “typical” markets

Connection between economic, algorithmic properties

Structure of Equilibrium Prices for GS

Recall: $(\mathcal{S}, p)$ is a Walrasian market equilibrium if:

- $\forall i$: S_i is in i's demand given p;
- the market clears

Fix GS market, let P be all equil. prices

Theorem: [Gul-Stacchetti'99] Equil. prices form a complete lattice

- If p, p' are equil. prices then so are $p \vee p', p \wedge p'$
- $\overline{p} = \bigvee P$ (component-wise $\sup$) and $\underline{p} = \bigwedge P$ (component-wise $\inf$) exist in P

Economic Characterization of Extremes

$\overline{p}$ = max. equil. price, $\underline{p}$ = min. equil. price

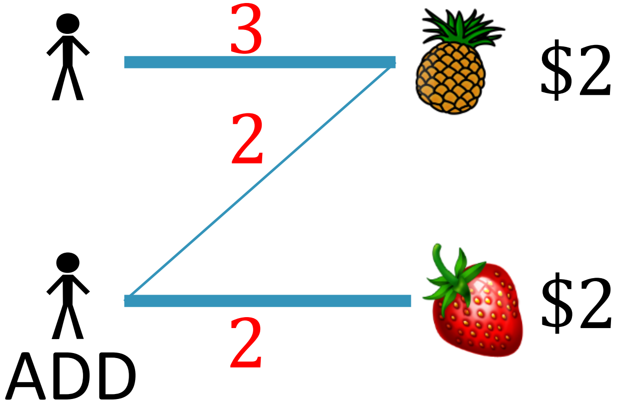
<u>Theorem</u>: [Gul-Stacchetti'99] In monotone GS markets,

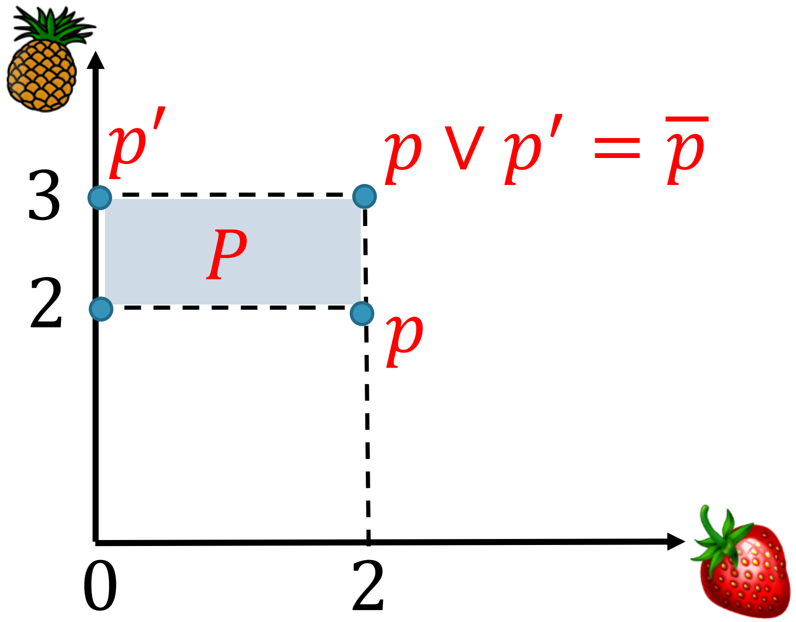
- $\overline{p}_j$ = decrease in welfare if j removed from the market
- $\underline{p}_j$ = increase in welfare if another copy (perfect substitute) of j added to the market

Example

Max. welfare is 5

- 2 with no pineapple, 3 with no strawberry
- 7 with extra pineapple, 5 with extra strawberry





A Corollary

$\underline{p}$ = min. equil. prices

$\underline{p}_j$ = welfare increase if copy of j is added to the market [GS'99]

In unit-demand markets, $\underline{p}$ coincides with VCG prices

- Let i be the player allocated j in VCG
- i pays for j the difference in welfare buyers $M \setminus \{i\}$ can get from N and from $N \setminus \{j\}$

Economic Implications of GS

✓ 1. Equilibrium prices form a lattice
2. VCG outcome monotone, in the core
3. Prices coordinate "typical" markets

VCG Auction

Multi-item generalization of Vickrey (2nd price) auction

The only dominant-strategy truthful, welfare-maximizing auction in which losers do not pay

But is it practical?

To analyze its properties let's define the coalitional value function w

Coalitional Value Function w

Definition:

w maps any coalition of players $C \subseteq G$ to the max. welfare from reallocating C's items among its members

- Without the seller (for $C: 0 \notin C$), $w(C) = 0$
- For the grand coalition, $w(G)$ = max. social welfare

(w immediately defines a cooperative game among the players – we'll return to this)

VCG Auction in Terms of w

w = coalitional value function

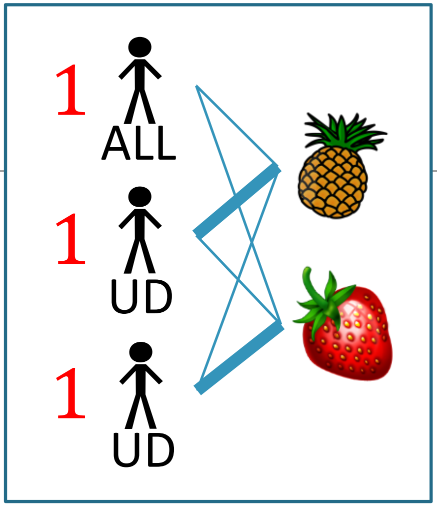
VCG allocation: Welfare-maximizing

VCG utilities: For every buyer $i > 0$,

$$\pi_i = w(G) - w(G \setminus \{i\})$$

(a buyer's utility is her marginal contribution to the social welfare; seller's utility is the welfare minus the marginals)

When VCG Goes Wrong



Example: 2 items

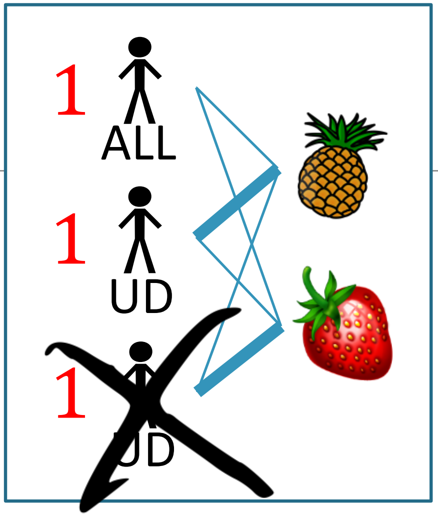
- Buyer 1: All-or-nothing with value 1
- Buyers 2 and 3: Unit-demand with value 1

VCG:

- Allocation: Buyers 2, 3 each get an item
- Utilities of players 0 to 3: $(0, 0, 1, 1)$

VCG outcome blocked by coalition of players 0 and 1! 😱

When VCG Goes Wrong



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- Buyer 1: All-or-nothing with value 1
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VCG:

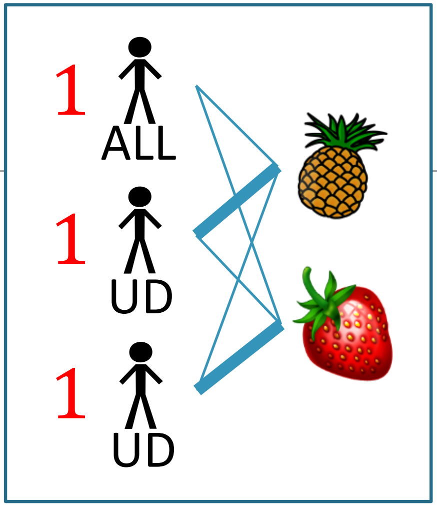
- Allocation: Buyers 2, 3 each get an item
- Utilities of players 0 to 3: $(0, 0, 1, 1)$

VCG without buyer 3:

- Allocation: Buyer 2 gets as item (or buyer 1 gets both)
- Utilities of players 0 to 2: $(1, 0, 0)$

Non-monotone revenue! 😱

What Goes Wrong



VCG:
- Utilities of players 0 to 3: $(0, 0, 1, 1)$

VCG without buyer 3:
- Utilities of players 0 to 2: $(1, 0, 0)$

Buyers 2's marginal contribution to the welfare increases when the coalition includes buyer 3

→ coalitional value function w is not submodular

Characterization of Good VCG

w = coalitional value function

$\pi(C)$ = utility profile from applying VCG to coalition C

Theorem [Ausubel-Milgrom'02]: Equivalence among -

1. For every C, $\pi(C)$ is in the core (not blocked by any coalition)
2. For every C, $\pi(C)$ is monotone in buyers
   - in particular, revenue-monotone
3. Function w is buyer-submodular
   - (= submodular when restricted to coalitions including the seller)

Buyer-Submodularity and GS

w = coalitional value function

$\mathcal{V}$ = class of valuations that contains additive valuations

Theorem [Ausubel-Milgrom'02]:

For w to be buyer-submodular for every market with valuations $\subseteq \mathcal{V}$, a necessary and sufficient condition is that $\mathcal{V} \subseteq$ GS

"Maximal domain" result

Proof Sketch: Sufficiency

Recall: For GS markets, the maximum welfare is equal to

$$\min_p \{\sum_{i\in M} u_i(p) + p(N)\}$$

where $u_i(\cdot)$ = Fenchel dual

Applied to buyer coalition $C \subseteq M$,

$$w(C \cup \{0\}) = \min_p \{\sum_{i\in C} u_i(p) + p(N)\}$$

Proof Sketch: Sufficiency

$$w(C \cup \{0\}) = \min_p \{\underbrace{\textstyle\sum_{i \in C} u_i(p) + p(N)}_{\text{Denote by } f(p,C)}\}$$

Since Fenchel duals $\{u_i\}$ are submodular on $\mathbb{R}^n$ for GS

→ f is submodular on the product lattice $\mathbb{R}^n \times 2^M$

A result by [Topkis'78] shows $\min_p\{f(p,C)\}$ is submodular on 2^M.

QED

*Based on slides by Paul Milgrom

Proof Sketch: Necessity

Let v be non-GS

Consider a coalition of v with additive valuation p':

$$w(\{v, p'\}) =$$

$$\min_p \{u(p) + \sum_j \max\{0, p'_j - p_j\} + p(N)\} =$$

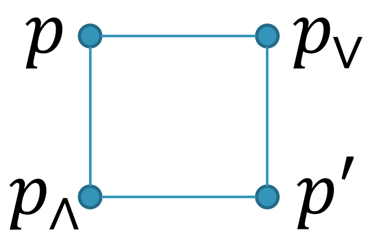
$$u(p') + p'(N)$$

- Generalizes to coalitions with several additive valuations by observing their join is the minimizer

*Based on slides by Paul Milgrom

Proof Sketch: Necessity

Let v be non-GS → Fenchel dual u non-submodular



$$\exists p, p' : u(p_\vee) + u(p_\wedge) > u(p) + u(p')$$

Add 3 additive buyers with valuations $p, p', p_\wedge$

$$w(\{v, p_\wedge\}) = u(p_\wedge) + p_\wedge(N)$$
$$w(\{v, p_\wedge, p, p'\}) = u(p_\vee) + p_\vee(N)$$
$$>$$
$$w(\{v, p_\wedge, p\}) = u(p) + p(N)$$
$$w(\{v, p_\wedge, p'\}) = u(p') + p'(N)$$

→ w not buyer-submodular. QED

*Based on slides by Paul Milgrom

Economic Implications

✓ 1. Equilibrium prices form a lattice
✓ 2. VCG outcome monotone, in the core
3. Prices coordinate "typical" markets

Breather

Riddle: How is Fenchel connected to the building below?

- German-born Jewish mathematician who emigrated following Nazi suppression and settled in Denmark
- His younger brother Heinz immigrated to Israel and became a renowned architect, designing this Tel-Aviv landmark

Do Equil. Prices Coordinate Markets?

Question posed by [Hsu+'16], following [Hayek'45]:

- *"Fundamentally, in a system in which the knowledge of the relevant facts is dispersed among many people, prices can act to coordinate the separate actions of different people…"*

The American Economic Review

VOLUME XXXV SEPTEMBER, 1945 NUMBER FOUR

THE USE OF KNOWLEDGE IN SOCIETY

By F. A. HAYEK*

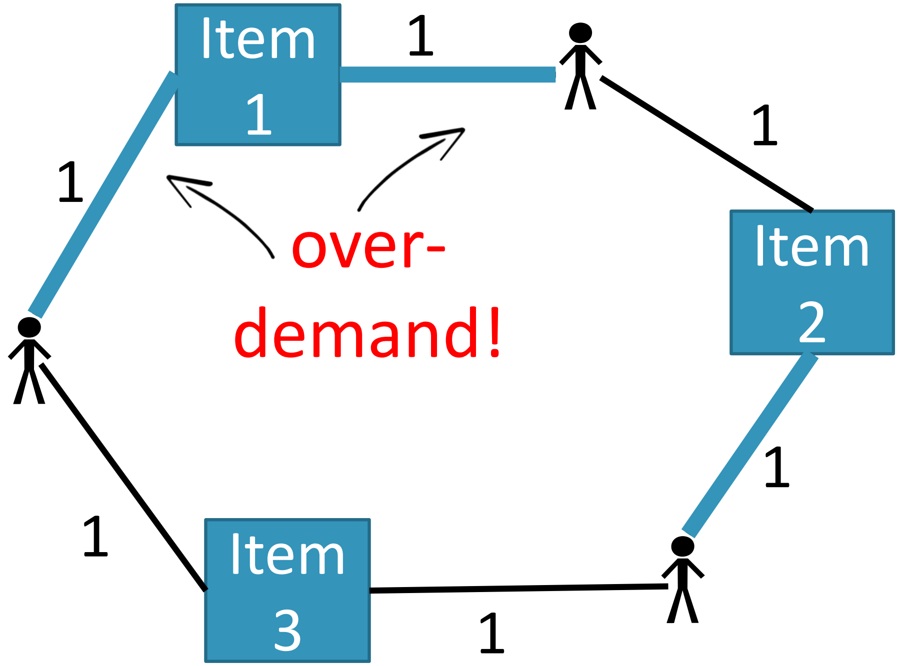
I

What is the problem we wish to solve when we try to construct a rational economic order?

On certain familiar assumptions the answer is simple enough. *If* we possess all the relevant information, *if* we can start out from a given

Bad Example with GS Valuations

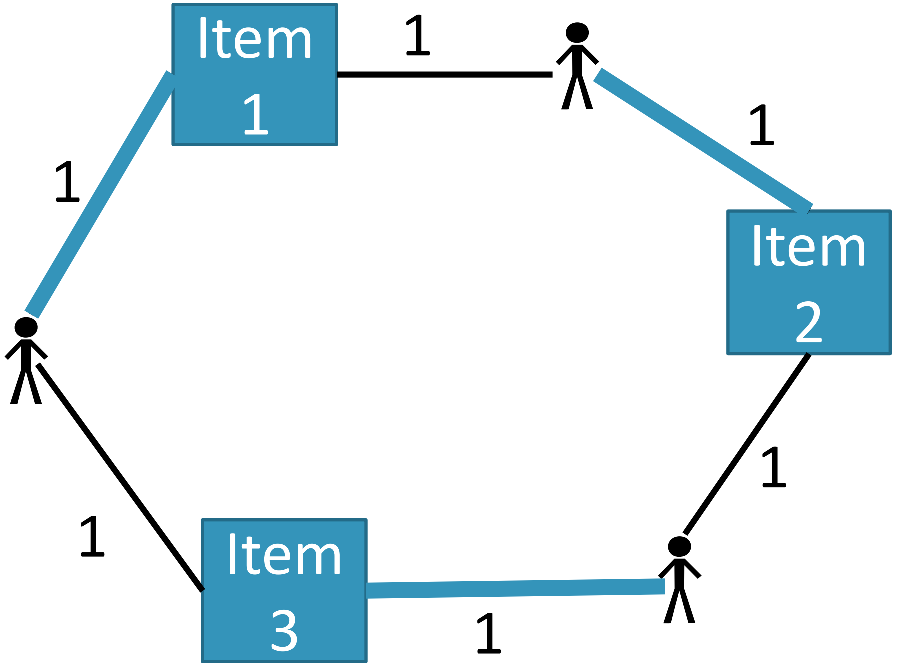
[Cohen-Addad-et-al'16]: Wlog $p_1 \leq p_2 \leq p_3$



*Based on slides by Alon Eden

What Goes Wrong

Welfare-maximizing allocation is not unique



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What Goes Wrong

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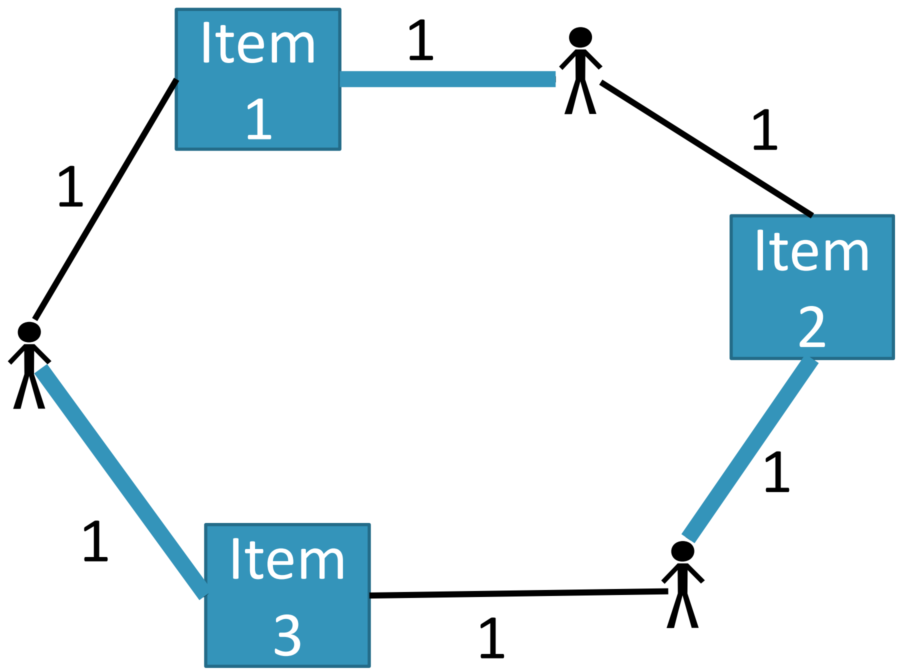

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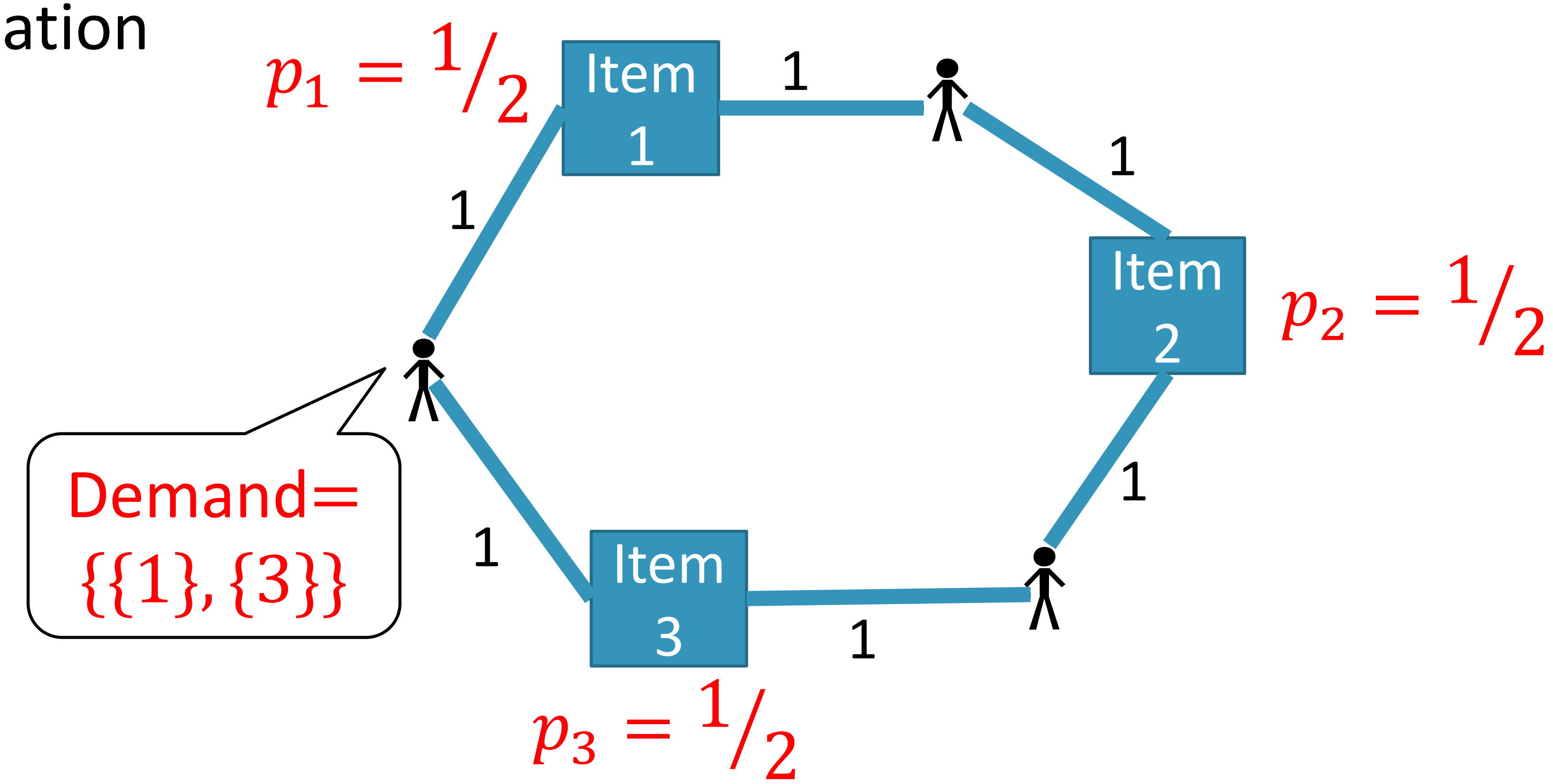
Uniqueness Necessary for Coordination

By 2nd Welfare Theorem: Equilibrium prices support any max-welfare allocation



Coordinating Prices

Definition:

Walrasian equilibrium prices p are robust if every buyer has a single bundle in demand given p

- Robust prices are market-coordinating

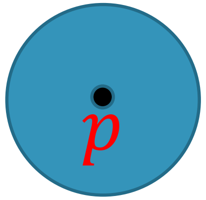
Theorem: [Cohen-Added-et-al'16, Paes Leme-Wong'17]

For a GS market, uniqueness of max-welfare allocation is sufficient for existence of robust equil. prices

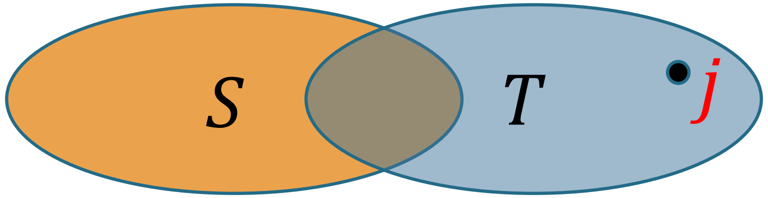
- Moreover, almost all equil. prices are robust

Pf: Uniqueness is Sufficient

Plan: Assume GS + uniqueness of max-welfare allocation (and integral values for simplicity); show a ball of equilibrium prices exists

This establishes robust pricing: 

Assume for contradiction both S^*, T in player's demand given p



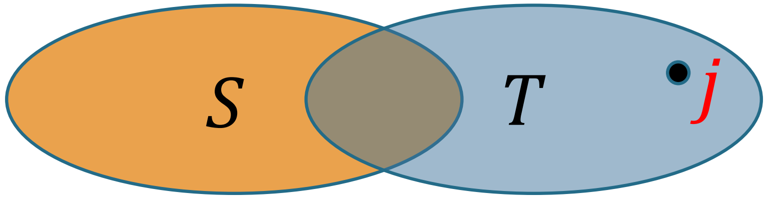
Pf: Uniqueness is Sufficient

Plan: Assume GS + uniqueness of max-welfare allocation (and integral values for simplicity); show a ball of equilibrium prices exists

This establishes robust pricing: 

Assume for contradiction both S^*, T in player's demand given p

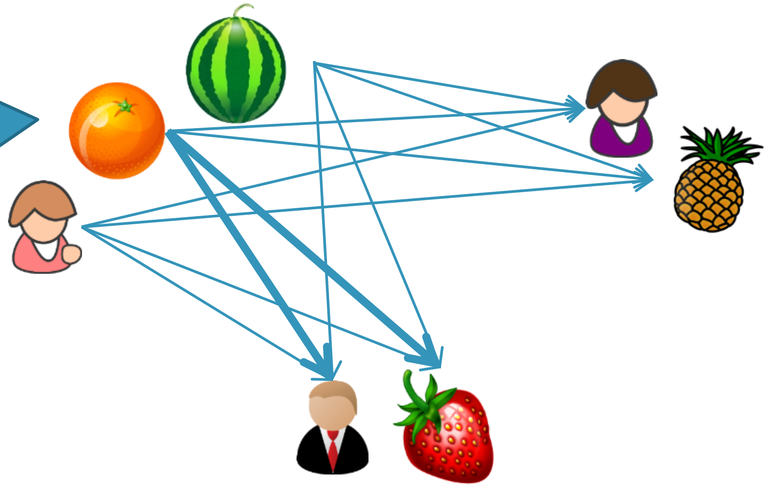
Let $p' = p$ with p_j decreased; should also support S^*, contradiction



Pf: Exchange Graph [Murota]

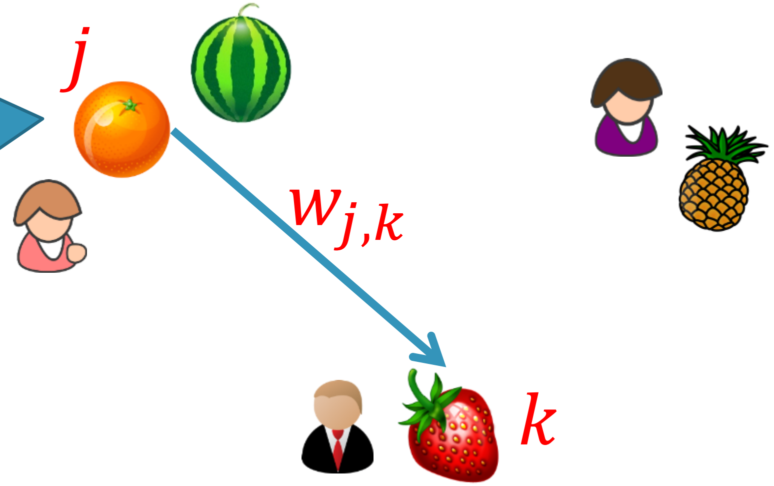
Exchange graph for the unique max-welfare allocation:

Edge weights w = how much buyer would lose from exchanging orange with strawberry (or giving up orange)



Pf: Cycles and Equilibrium Prices

Edge weights w = how much buyer would lose from exchanging orange with strawberry (or giving up orange)

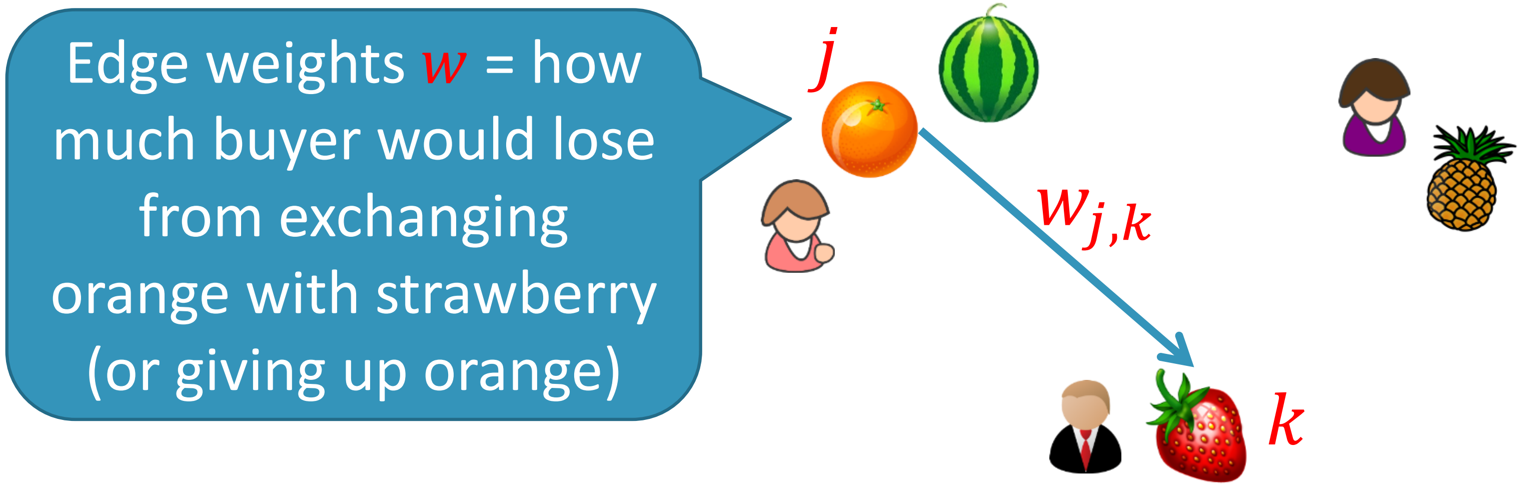


A function ϕ on the nodes is a potential if $w_{j,k} \geq \phi(k) - \phi(j)$

Theorem: $\exists$ potential $\phi \Leftrightarrow$ no negative cycle $\Leftrightarrow -\phi$ = equil. prices

Theorem: $\exists$ ball of potentials / equil. prices $\Leftrightarrow$ all cycles strictly positive

Pf: Ball of Equilibrium Prices



Theorem: $\exists$ ball of equil. prices $\Leftrightarrow$ all cycles strictly positive

0-weight cycle = alternative max-welfare allocation. QED

Do Prices Coordinate Typical Markets?

I.e., do GS markets typically have a unique max-welfare allocation?

We say a GS market typically satisfies a condition if it holds whp under a tiny random perturbation of arbitrary GS valuations

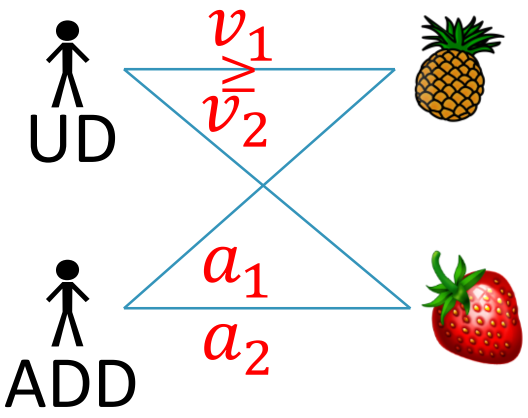
Challenge: Find a perturbation model that maintains GS

- (Ideally one in which the perturbation can be drawn from a discrete set)

2 GS-Preserving Perturbation Models

For simplicity, unit-demand v_i

The perturbation: additive valuation a_i



1. $v_i'(S) = v_i(S) + a_i(S)$ [P-LW'17]
   - v_i' not unit-demand
2. $v_i'(j) = v_i(j) + a_i(j)$ [Hsu+'16]
   - v_i' unit-demand

1. $v_i'(N) = v_1 + a_1 + a_2$
2. $v_i'(N) = v_1 + a_1$

Unique Max-Welfare Allocation is Typical

Lemma: [P-LW'17,Hsu+'16]

For sufficiently small perturbation, whp the perturbed market has a unique max-welfare allocation

- (Also max-welfare in the original market)
- Perturbation can be from sufficiently large discrete range [*cf.* MVV'87 Isolation Lemma]

Market Coordination: Additional Results

[Cohen-Addad-et-al’16]: “Necessity” of GS for market coordination

- ∃ non-GS market with:

1. unique max-welfare allocation
2. Walrasian equilibrium
3. no coordinating prices (not even dynamic!)

[Hsu-et-al’16]: Robustness of min. equilibrium prices (not in ball)

- For perturbed markets such prices induce little overdemand

Economic Implications

✓ 1. Equilibrium prices form a lattice
✓ 2. VCG outcome monotone, in the core
✓ 3. Prices coordinate "typical" markets

Recap

GS plays central role in the following:

1. Equilibrium prices exist and form a lattice
2. VCG outcome monotone, in the core
   - A GS market is characterized by a submodular coalitional value function w
   - Buyers' utilities in VCG are their marginal contribution to w
3. Prices coordinate "typical" markets
   - For GS, prices coordinate iff max-welfare allocation is unique
   - Perturbed GS markets have a unique max-welfare allocation

Necessity of GS Algorithmic Properties

Part I: Algorithmic properties of GS
- Frontier of tractability for DEMAND and WELFARE-MAX

Part II: Economic implications of GS
- Including existence of equil. prices

Is there a direct connection?

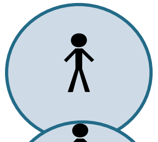
[RoughgardenT'15]: A direct connection between market equilibrium (non)existence and computational complexity of DEMAND, WELFARE-MAX

Market Equilibrium & Related Problems

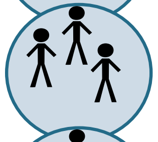
Recall: $(\mathcal{S}, p)$ is a Walrasian market equilibrium if:

$\forall i$: S_i solves DEMAND(v_i, p);
$\mathcal{S}$ solves REVENUE-MAX(p)

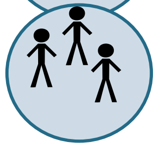
Related computational problems: $\mathcal{V}$ = class of valuations



DEMAND: On input $v \in \mathcal{V}$ and p, output a bundle S in demand given p



WELFARE-MAX: On input $v_1, \ldots, v_m \in \mathcal{V}$, output a max-welfare allocation $\mathcal{S}$



REVENUE-MAX: On input p, output a max-revenue allocation $\mathcal{S}$

From Complexity to Equil. Nonexistence

$\mathcal{V}$ = class of valuations

Theorem: [RoughgardenT'15]

- A necessary condition for guaranteed existence of Walrasian equil. for $\mathcal{V}$: DEMAND is at least as computationally hard as WELFARE-MAX for $\mathcal{V}$
- → If under $\mathrm{P} \neq \mathrm{NP}$ WELFARE-MAX is harder than DEMAND, equil. existence not guaranteed for $\mathcal{V}$

Example

$\mathcal{V}$ = capped additive valuations

DEMAND = KNAPSACK → pseudo-polynomial time algo.

WELFARE-MAX = BIN-PACKING → strongly NP-hard

If $\mathrm{P} \neq \mathrm{NP}$ then WELFARE-MAX is harder than DEMAND

<u>Conclusion</u>: equil. existence not guaranteed for capped additive

Complexity Approach: Some Pros & Cons

Con: Need $\mathrm{P} \neq \mathrm{NP}$ (or similar) assumption

Pros: Alternative to "maximal domain" results

- Case in point: Equil. existence not guaranteed for $\mathcal{V}$: unit-demand $\subseteq \mathcal{V}$ unless $\mathcal{V} =$ GS [GulStacchetti'99]
- Misses many $\mathcal{V}$s that do not contain unit-demand

Gross substitutes and complements [Sun-Yang'06, Teytelboym'13], k-gross substitutes [Ben-Zwi'13], superadditive [Parkes-Ungar'00, Sun-Yang'14], tree, graphical or feature-based valuations [Candogan'14, Candogan'15, Candogan-Pekec'14], …

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The complexity approach generalizes to show nonguaranteed existence of relaxed equilibrium notions in typical markets

Open direction: Apply the complexity approach to other economic properties of GS

2. Pushing the Boundaries of GS

- ROBUSTNESS OF THE ALGORITHMIC PROPERTIES
- EXTENDING THE ECONOMIC PROPERTIES

Motivation: Incentive Auction Mystery

"Few FCC policies have generated more attention than the Incentive Auction.

'Groundbreaking,' 'revolutionary,' and 'first-in-the-world' are just a few common descriptions of this innovative approach to making efficient, market-driven use of our spectrum resources."

- $20 billion auction
- Freed up 84 MHz of spectrum
- 2018 Franz Edelman Award

Incentive Auction Model

TV broadcasters with values $v_1, \dots, v_m$ for staying on the air

- Auction outcome = on-air broadcaster set A
- A repacked into a reduced band of spectrum

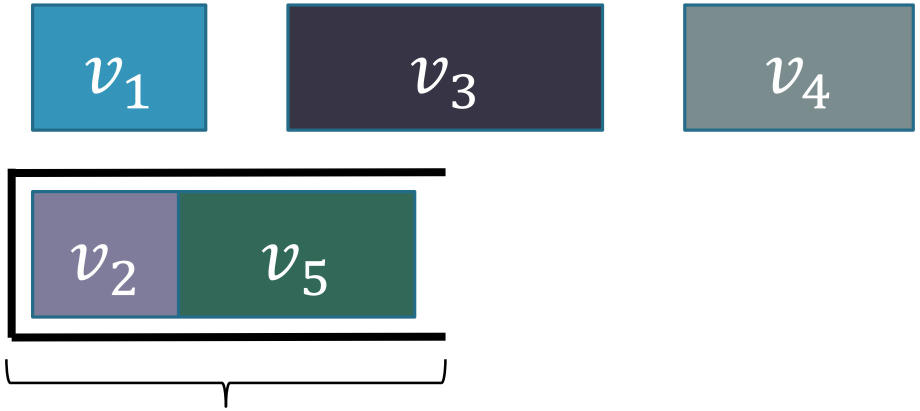
Feasibility constraint:

- $\mathcal{F} \subseteq 2^M$ = sets of broadcasters that can be feasibly repacked
- Outcome is feasible if $A \in \mathcal{F}$
- $\mathcal{F}$ downward-closed

Incentive Auction Model

Broadcasters going off the air:

A = on-air broadcasters :



packing constraint (e.g. knapsack)

Goal: Minimize total value that goes off the air = maximize A's total value, subject to feasibility of repacking

$$\max_{A \in \mathcal{F}} \sum_{i \in A} v_i$$

The Mystery

Fact 1: $\max_{A \in \mathcal{F}} \sum_{i \in A} v_i$ greedily solvable iff $\mathcal{F}$ defines a matroid over the broadcasters

Equivalently, if v is GS where $v(A) = \max_{A \supseteq A' \in \mathcal{F}} \sum_{i \in A'} v_i$

Fact 2: In the Incentive Auction $\mathcal{F}$ is not a matroid

Fact 3: In simulations Greedy achieves $> 95\%$ of OPT on average

- Over values sampled according to FCC predictions
- [Newman, Leyton-Brown, Milgrom & Segal 2017]

The Mystery

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Is v "95% GS"?

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Is $\mathcal{F}$ "95% a matroid"?

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Research Agenda

In theory: Only GS markets guaranteed to work

Good algorithmic, economic properties

Folklore belief:

- Many markets work well in practice since they're "approximately GS"
- I.e. good properties are robust

Agenda: We need theory predicting when markets actually work well

- Starting with good models of "approximately GS"
- *Cf*. "beyond worst case" agenda (replace markets with algorithms…)

Plan

2 recent approaches to "approximate GS"

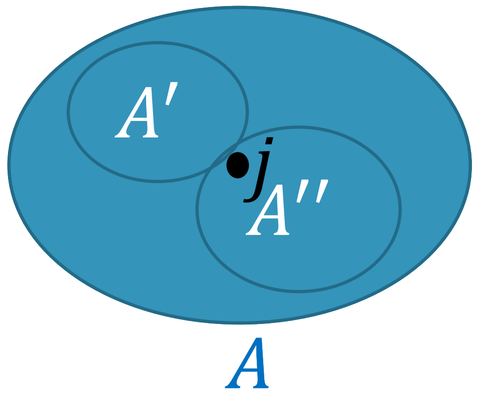
1. Start from good performance of greedy
2. Start from approximating a very basic GS subclass: linear valuations

Approach 1: Matroids

$\mathcal{F}$ defines a matroid over M if:

1. Rank quotient of $\mathcal{F}$ is 1

$$\min_{A \subseteq M} \min_{A', A'' \subseteq A \text{ maximal in } \mathcal{F}} \frac{|A'|}{|A''|} = 1$$



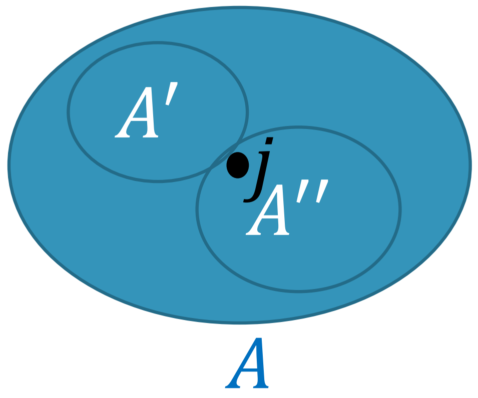
2. [Equivalently] The exchange property holds:
   - For every 2 feasible sets A', A'', if $|A'| < |A''|$ then there's an element we can add from A'' to A' while maintaining feasibility

Approximate Matroids

$\mathcal{F}$ defines a ρ-matroid over M for ANY $\rho \leq 1$ if:

1. Rank quotient of $\mathcal{F}$ is ρ

$$\min_{A \subseteq M} \min_{A', A'' \subseteq A \text{ maximal in } \mathcal{F}} \frac{|A'|}{|A''|} = \rho$$



2. [Equivalently] The ρ-exchange property holds:
   - For every 2 feasible sets A', A'', if $|A'| < \rho |A''|$ then there's an element we can add from A'' to A' while maintaining feasibility

Approximate Matroids

<u>Theorem</u> [Korte-Hausmann'78]:

$\max_{A\in\mathcal{F}} \sum_{i\in A} v_i$ greedily ρ-approximable for any values $v_1, \ldots, v_m$ iff $\mathcal{F}$ defines a ρ-matroid over M

<u>Note</u>: Recent alternative notion of approx. matroids [Milgrom'17]

- $\mathcal{F}$ is ρ-close to a matroid $\mathcal{M}$ if feasible sets in $\mathcal{F}$ ρ-covered by sets in $\mathcal{M}$
- Greedily optimizing wrt $\mathcal{M}$ gives ρ-approximation wrt $\mathcal{F}$

Open Questions

1. Does GS theory (approx.) extend to approx. matroid valuations?
   - Rank functions $v(A) = \max_{A \supseteq A' \in \mathcal{F}} \sum_{i \in A'} v_i$, and their closure under mergers etc.
2. Alternative approximation notions
   - E.g., which notion ensures that greedy approximately minimizes the total value going off the air
3. Empirical study
   - Is $\mathcal{F}$ in the Incentive Auction an approx. matroid?

Approach 2:

Study natural approximations of linear valuations

$$v(S) = v(\emptyset) + \sum_{j \in S} v(j) \text{ for all } S$$

Why linear?

- Fundamental but still many open questions
- Equivalent to modular

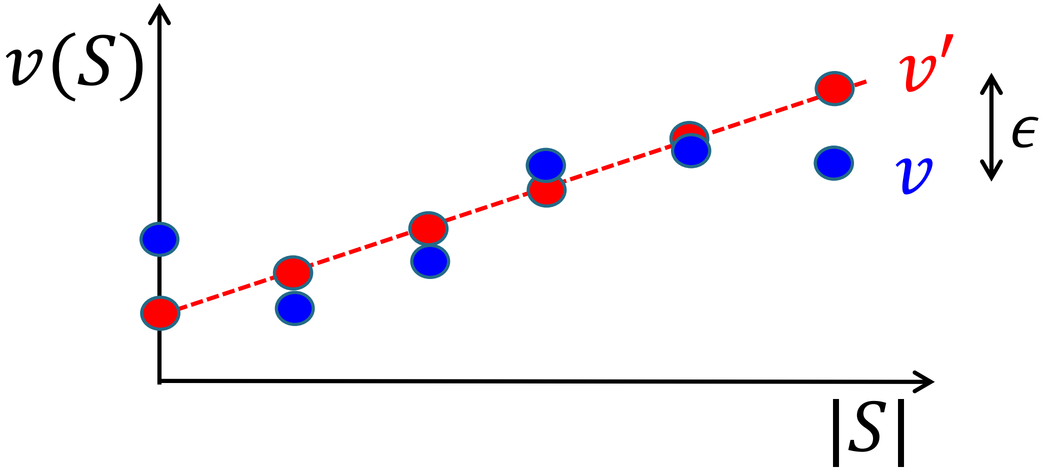
$$v(S) + v(T) = v(S \cup T) + v(S \cap T) \text{ for all } S, T$$

- Additive valuations ($v(\emptyset) = 0$) “too easy”

Natural Approximations of Linear

Pointwise approximation of linear v':

- Multiplicatively: $v'(S) \leq v(S) \leq (1+\epsilon)v'(S)$ for every S
- Additively: $|v'(S) - v(S)| \leq \epsilon$ for every S



Natural Approximations of Linear

Pointwise approximation of linear v':

- Multiplicatively: $v'(S) \leq v(S) \leq (1+\epsilon)v'(S)$ for every S
- Additively: $|v'(S) - v(S)| \leq \epsilon$ for every S

Approximate modularity: $|v(S) + v(T) - v(S \cup T) + v(S \cap T)| \leq \epsilon$

What's Known

v = pointwise (multiplicative) $(1+\epsilon)$-approximation of linear v'

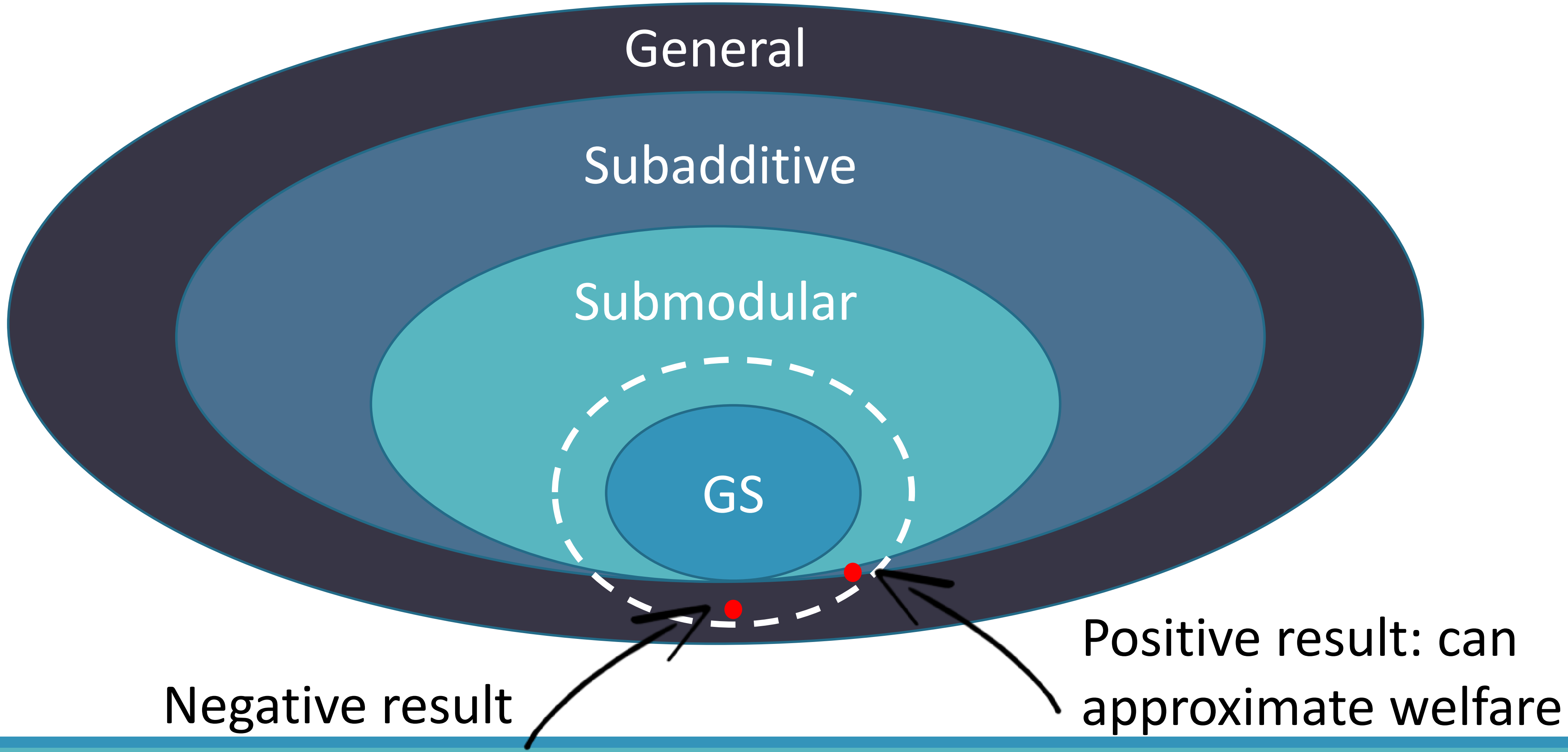
Theorem: [Roughgarden-T.-Vondrak'17]

- Without querying $v(S)$ exponentially many times, there is no const.-factor approximation of max. welfare

Unless v is also $(1+\alpha)$-approximately submodular

- Can get a $(1-3\epsilon)/(1+\alpha)$-approximation
- A la valuation hierarchies like $\mathcal{MPH}$ [FFIILS'15]

What's Known


General
Subadditive
Submodular
GS
Positive result: can approximate welfare
Negative result

What's Known

What about approximate modularity?

$$|v(S) + v(T) - v(S \cup T) + v(S \cap T)| \leq \epsilon$$

Theorem: [Feige-Feldman-T.'17]

- If v is ϵ-approximately modular then v is a pointwise (additive) 13ϵ-approximation of a linear v'

Summary

Incentive Auction mystery: Greedy works surprisingly well

Approaches:

1. Approx. matroids – needs more research
2. Approx. linear valuations – algorithmic properties not robust to natural approximation notions

Alternative Approaches

Other reasons why worst-case instances wouldn't appear in practice

Stable welfare-maximization [Chatziafratis et al.'17]

- Small changes in the valuations do not change max-welfare allocation
- Analog of "large margin" assumption in ML

Revealed preference approach [Echenique et al.'11]

- Data: (prices, demanded bundle) pairs
- For rationalizable data, there always exists a consistent tractable valuation

2. Pushing the Boundaries of GS

✓ ROBUSTNESS OF THE ALGORITHMIC PROPERTIES

- EXTENDING THE ECONOMIC PROPERTIES

Matching with Contracts

[Roth-Sotomayor'90] "Two-Sided Matching" book
- Separates models with and without money but shows similar results

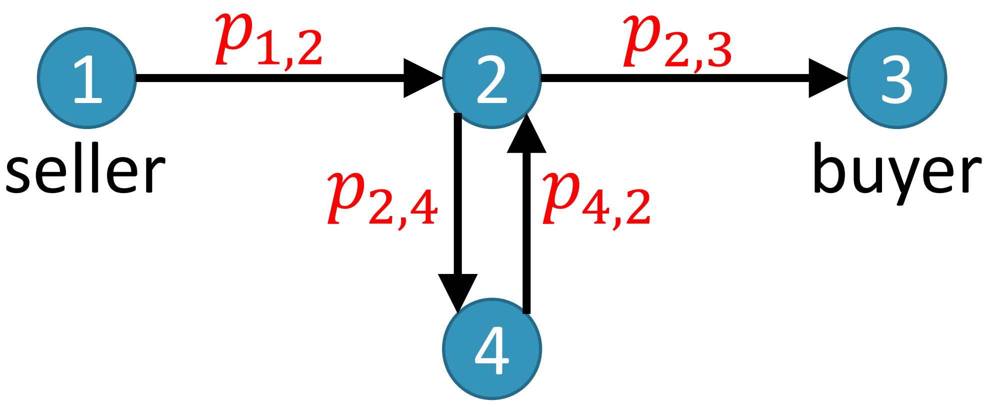
[Hatfield-Milgrom'95] "Matching with Contracts"
- Unifies the models (e.g., doctors and hospitals with combinatorial auctions)
- Bilateral "contracts" specify the matching and its conditions (like wages)
- Substitutability of the preferences plays an important role

[HKNOW'18] The most recent (?) in a long line of research
- Unifying different substitutability concepts for an individual agent
- Unifying stability and equilibrium concepts for markets

A General Model: Trading Networks

A multi-sided setting with:

- Nodes = agents (a buyer in some trades can be a seller in others)
- Directed edges = trades
- Valuations over set of trades, prices, quasi-linear utilities



$$\pi_4 = v_4(\{(2,4),(4,2)\}) - p_{2,4} + p_{4,2}$$

Trading Networks

Main results: Under substitutability of the valuations,

- Market equilibrium exists
- Equilibria equivalent to stable outcomes (i.e., cannot be blocked by coalitions of trades, where sufficient to consider paths/cycle)

[Candogan-Epitropou-Vohra'16] show equivalence to network flow

- Equilibria correspond to optimal flow and its dual
- Stability corresponds to no improving cycle
- Algorithmic implications

Demand Types [Baldwin-Klemperer'18]

New way of describing valuation classes

- Possible ways in which demand can change in response to small price change

Yields new characterization theorem for market equil. existence

<u>Example</u>:

- 2 items
- Class of unit-demand valuations
- Demand type:

$$\pm\{(1,-1),(0,1),(1,0)\}$$

Characterization Theorem

Theorem: [BK'18]

A market equilibrium exists for any market with concave valuations of demand type $\mathcal{D}$ iff $\mathcal{D}$ is unimodular

- Unimodular = every set of n vectors has a determinant 0, 1 or -1

Main Take Away

Much more to study in the realm of GS:

1. Recent fundamental results (like unique max-welfare allocation → price coordination)
2. Strong ties to algorithms (like trading networks vs. network flow, equil. existence vs. computational complexity) and math (like unimodularity thm)
3. Open crucial puzzles (like beyond worst case performance of greedy)

Some Related EC Talks

Tuesday@2:25PM **Combinatorial auctions with endowment effect**
Moshe Babaioff, Shahar Dobzinski and Sigal Oren

Tuesday@2:25PM **Designing core-selecting payment rules: A computational search approach**
Benjamin Lubin, Benedikt Bunz and Sven Seuken

Tuesday@2:25PM **Fast core pricing for rich advertising auctions**
Jason Hartline, Nicole Immorlica, Mohammad Reza Khani, Brendan Lucier and Rad Niazadeh

Thursday@2:10PM **Trading networks with frictions**
Tamas Fleiner, Ravi Jagadeesan, Zsuzsanna Janko and Alexander Teytelboym

Thursday@2:10PM **Chain stability in trading networks**
John Hatfield, Scott Kominers, Alexandru Nichifor, Michael Ostrovsky and Alexander Westkamp

Thursday@4PM **On the construction of substitutes**
Eric Balkanski and Renato Paes Leme

And more…