Gross Substitutes Tutorial

Part I: Combinatorial structure and algorithms
(Renato Paes Leme, Google)

Part II: Economics and the boundaries of substitutability
(Inbal Talgam-Cohen, Hebrew University)

Three seemingly-independent problems

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[Kelso-Crawford '82]
necessary /"sufficient"
condition for price
adjustment to converge

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Discrete Convex Analysis

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generalize convexity to discrete domains

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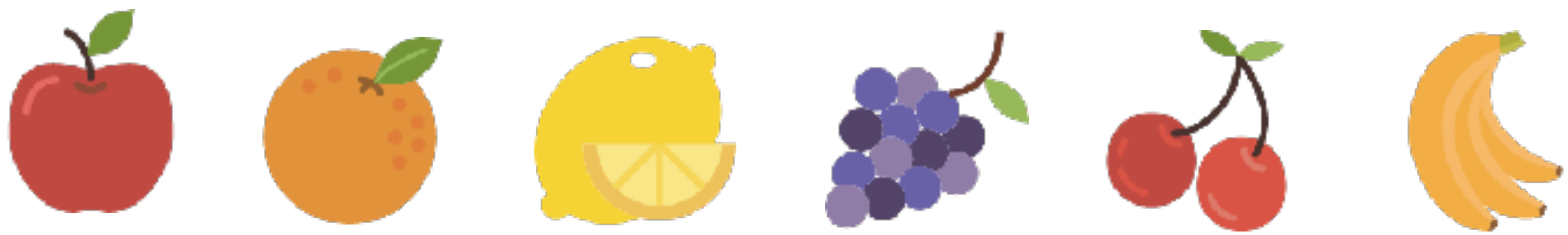
Some notation to start

- Discrete sets of goods: $[n] = \{1, \ldots, n\}$
- Valuation function $v : 2^{[n]} \to \mathbb{R}$
- Given prices $p \in \mathbb{R}^n$ define $v_p(S) = v(S) - p(S)$
- Demand correspondence $D(v; p) = \operatorname{argmax}_S v_p(S)$
- Demand oracle $\mathcal{O}_D(v, p) \in D(v; p)$
- Value oracle $\mathcal{O}_V(v, S) = v(S)$
- Marginals $v(S|T) = v(S \cup T) - v(T)$

Walrasian equilibrium

n goods



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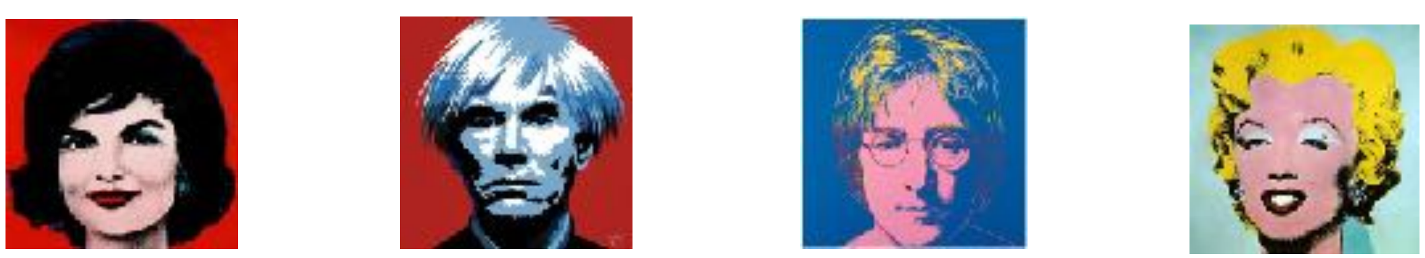

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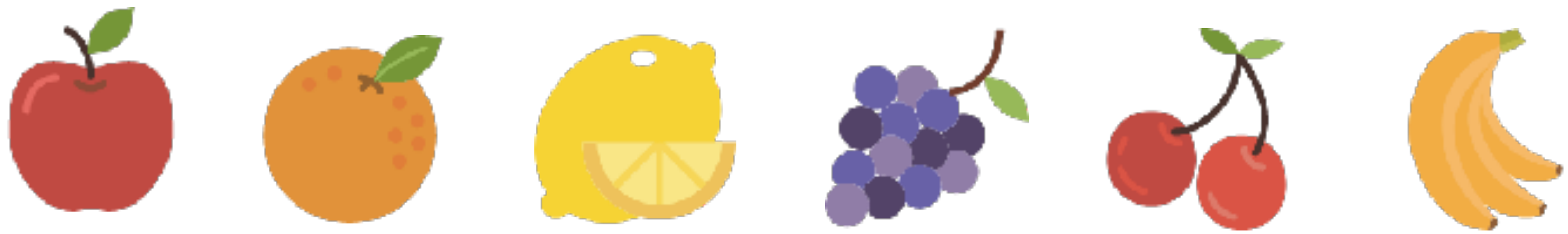

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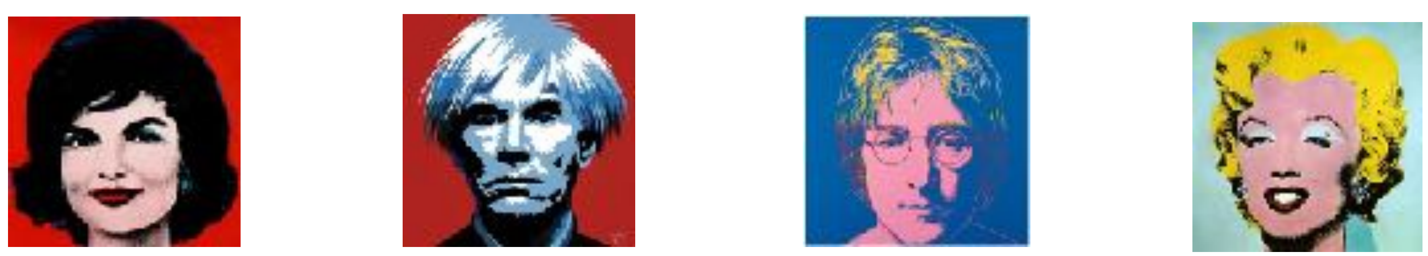

v_1 v_2 v_3 v_4

- Valuations $v_i : 2^N \to \mathbb{R}$

Walrasian equilibrium

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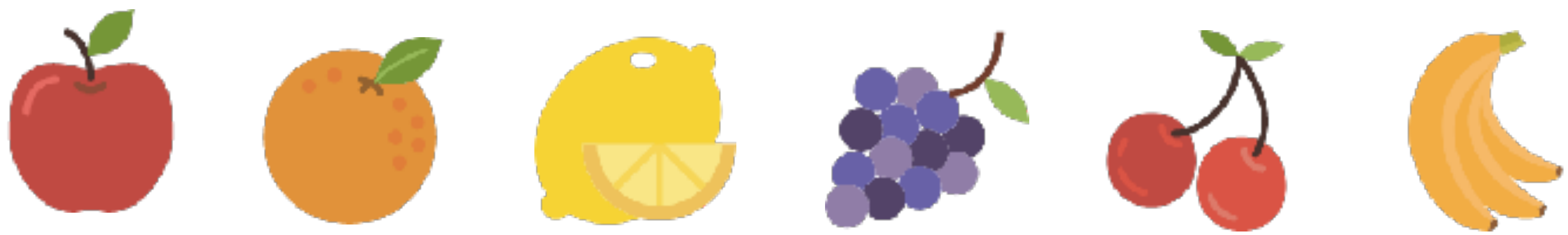

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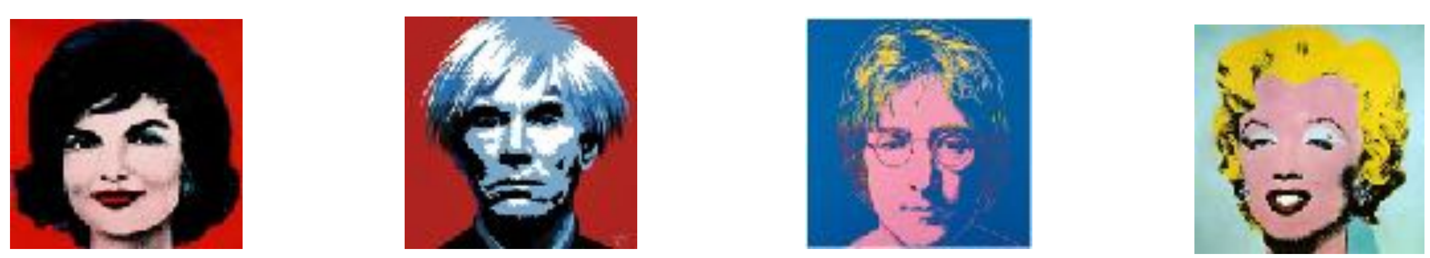

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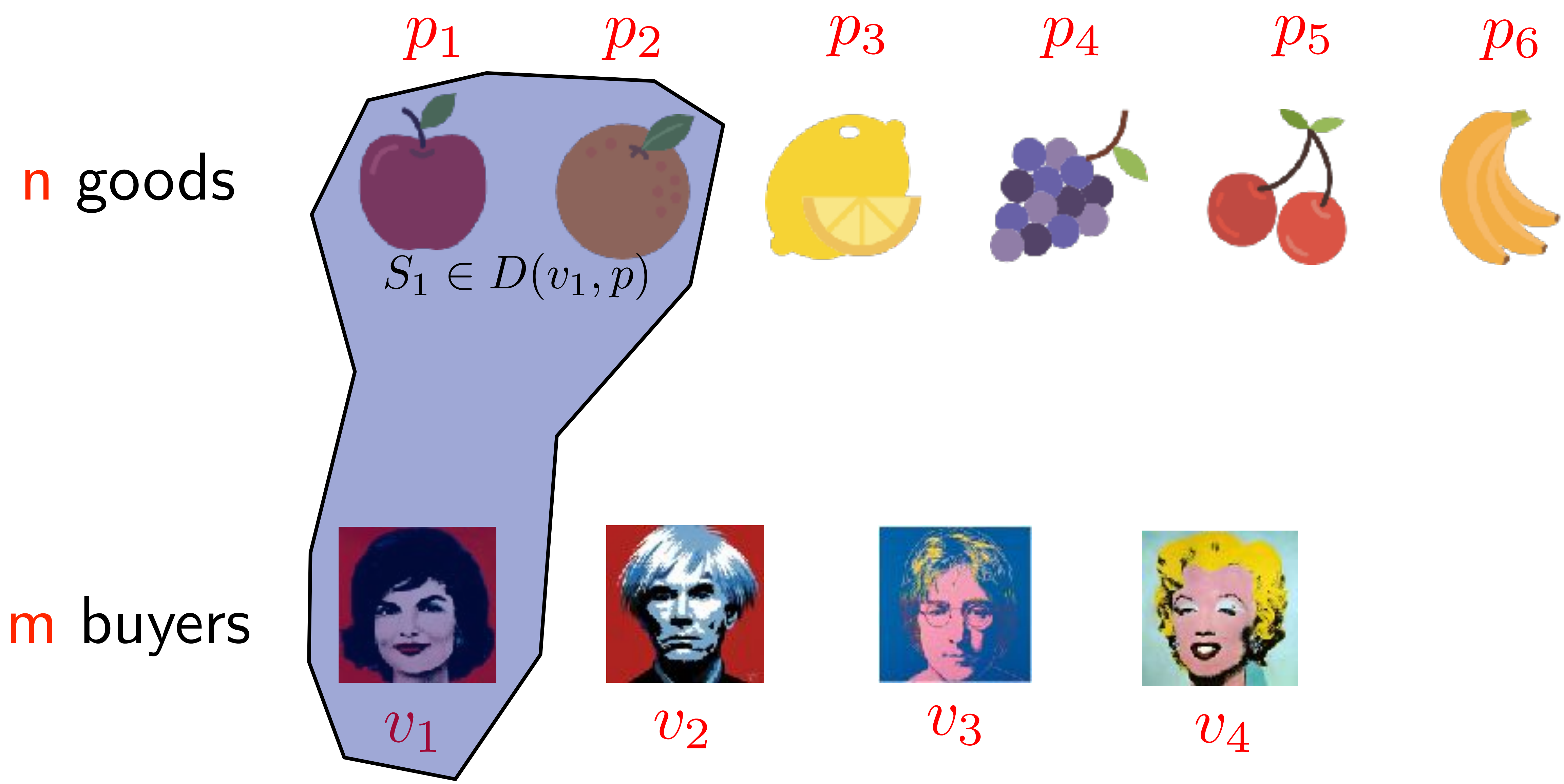

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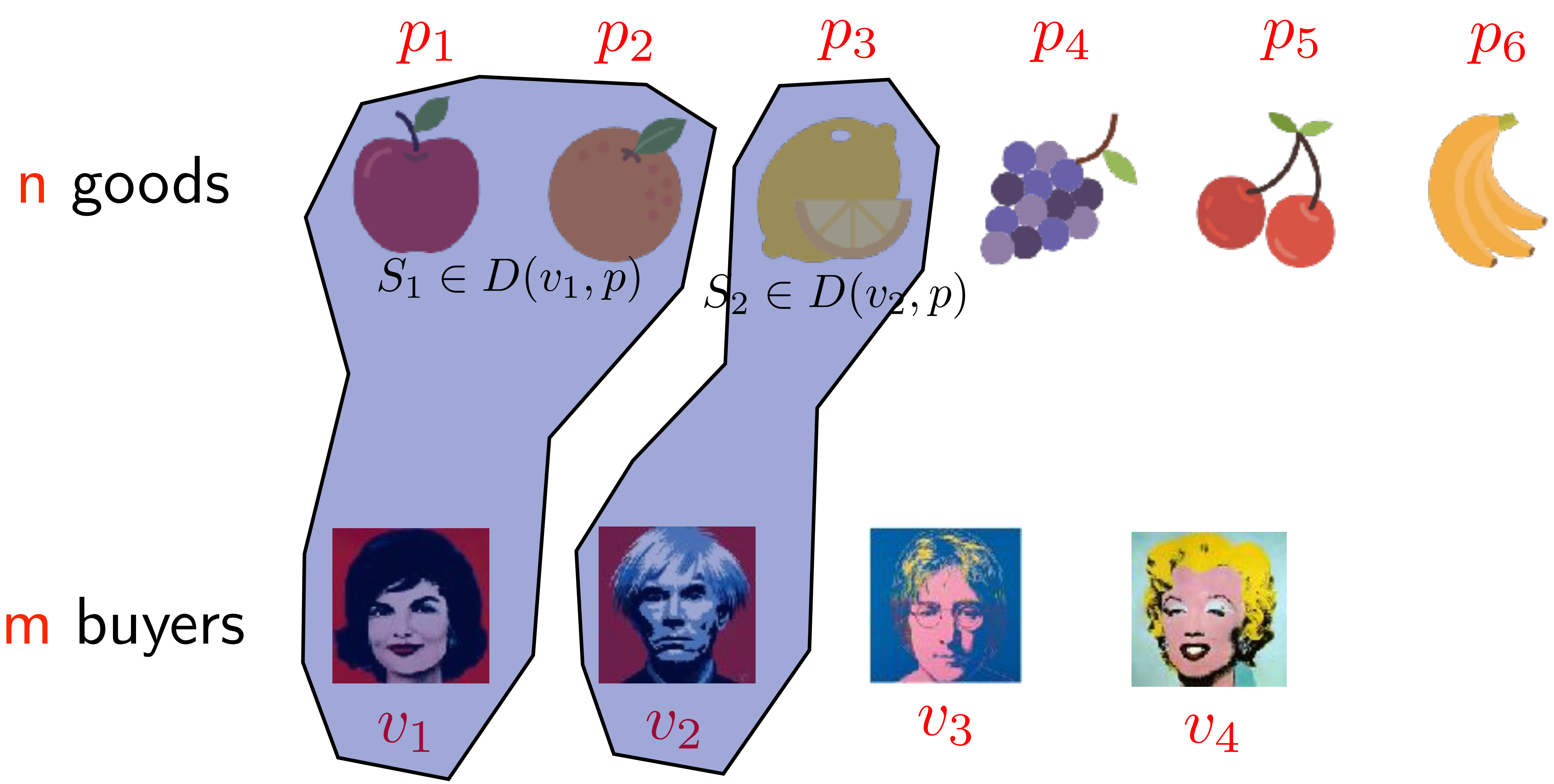

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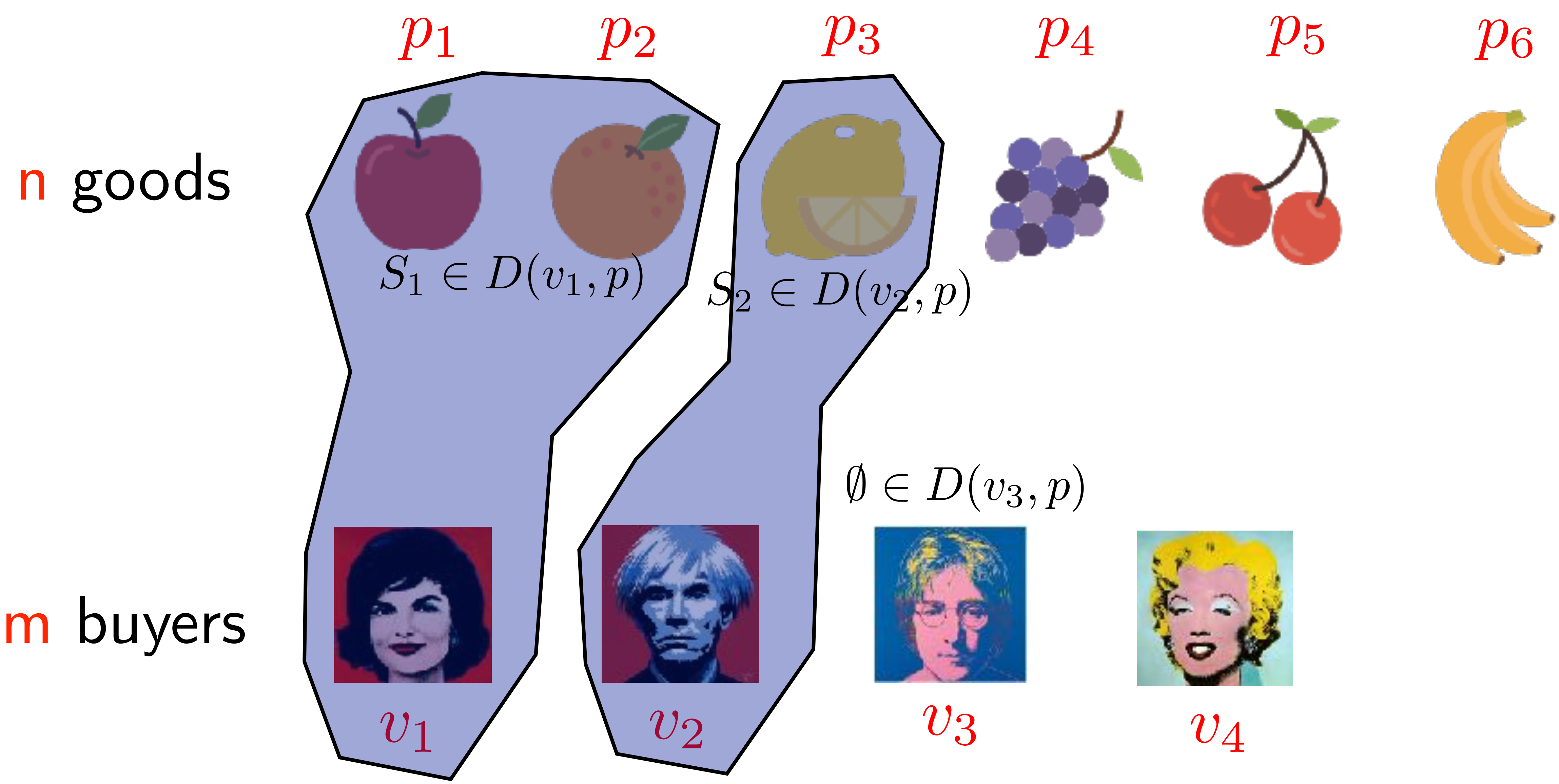

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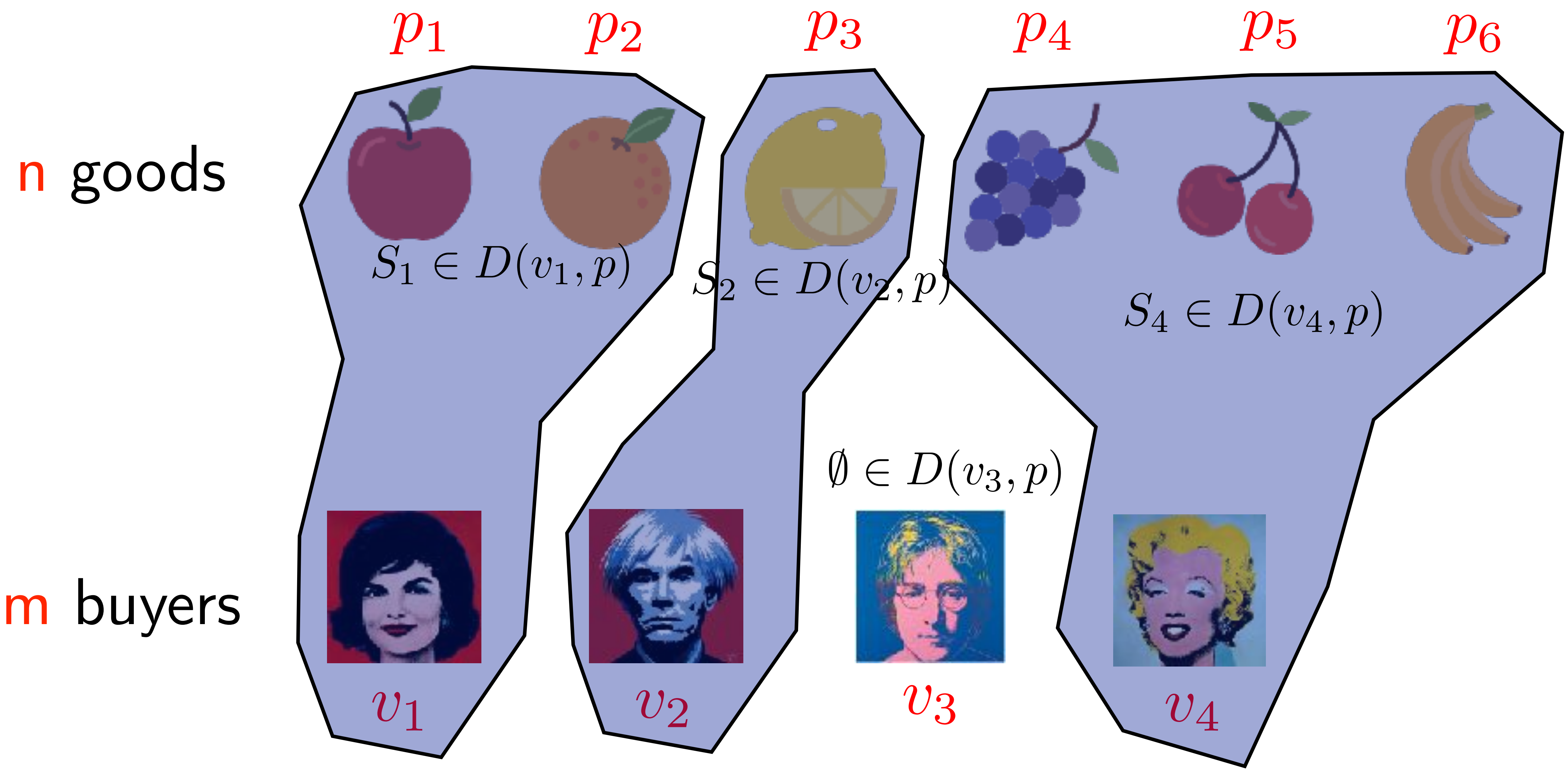

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- Market equilibrium: prices $p \in \mathbb{R}^n$ s.t. $S_i \in D(v_i, p)$ i.e. each good is demanded by **exactly** one buyer.

> **First Welfare Theorem**: in equilibrium the welfare $\sum_i v_i(S_i)$ is maximized.

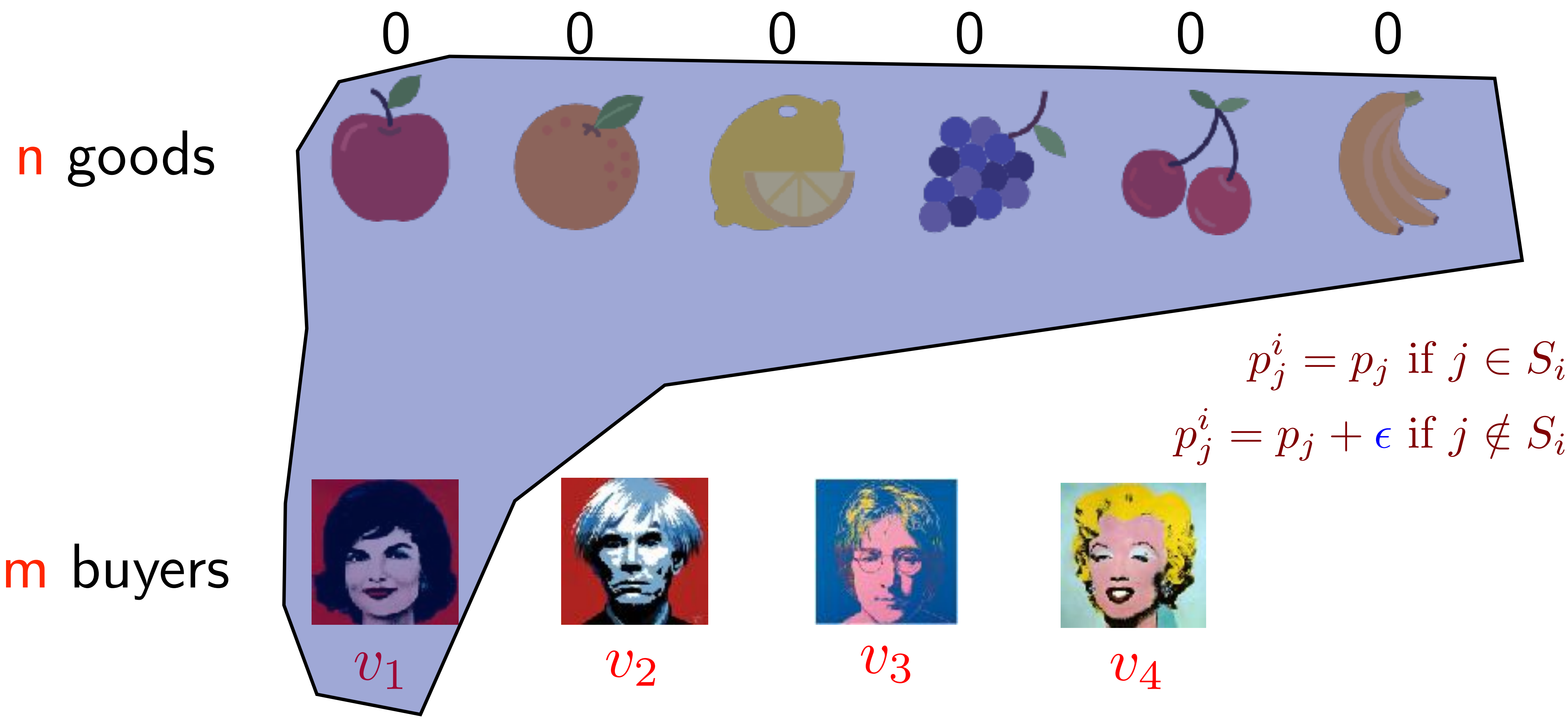
(proof: LP duality)

When do equilibria exist ?

How do markets converge to equilibrium prices ?

How to compute a Walrasian equilibrium ?

Walrasian tatonnement



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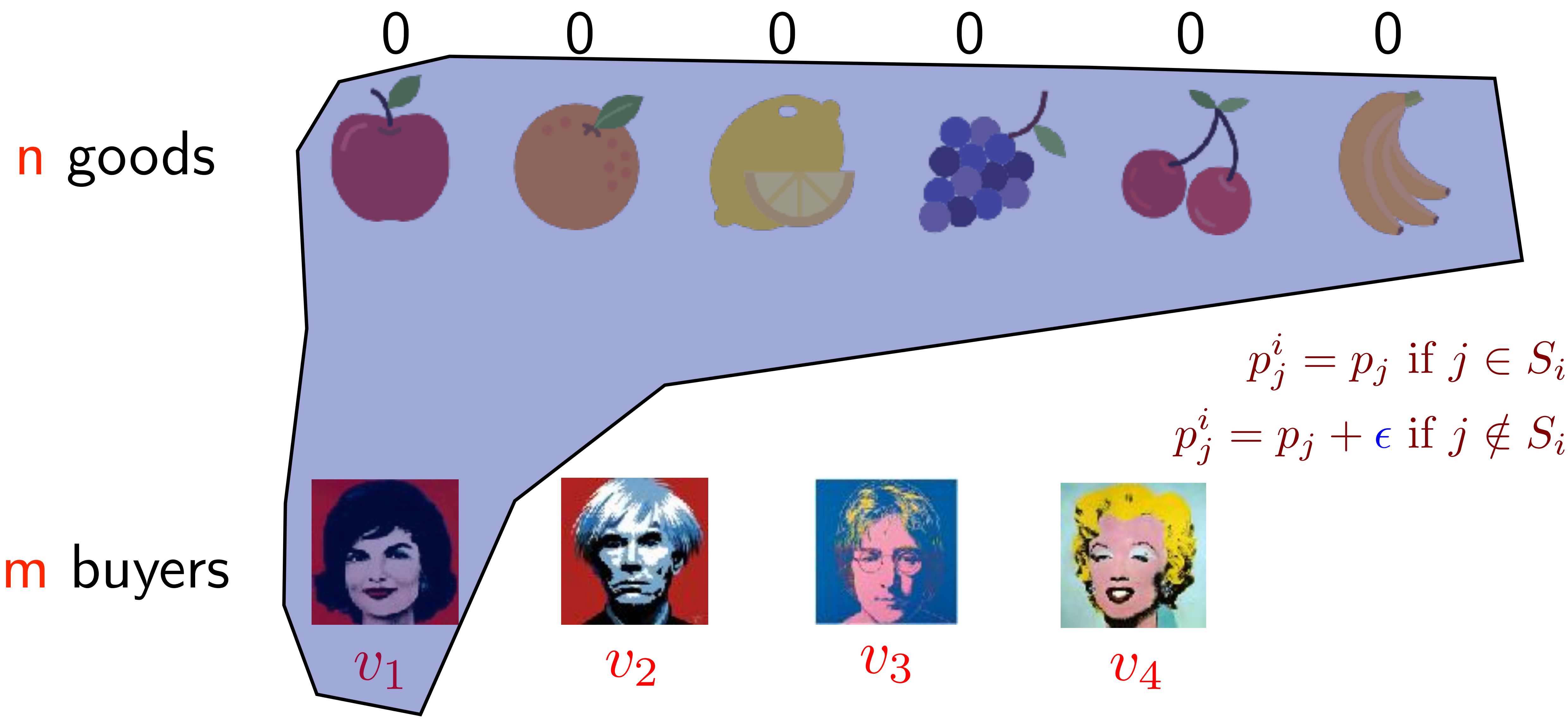

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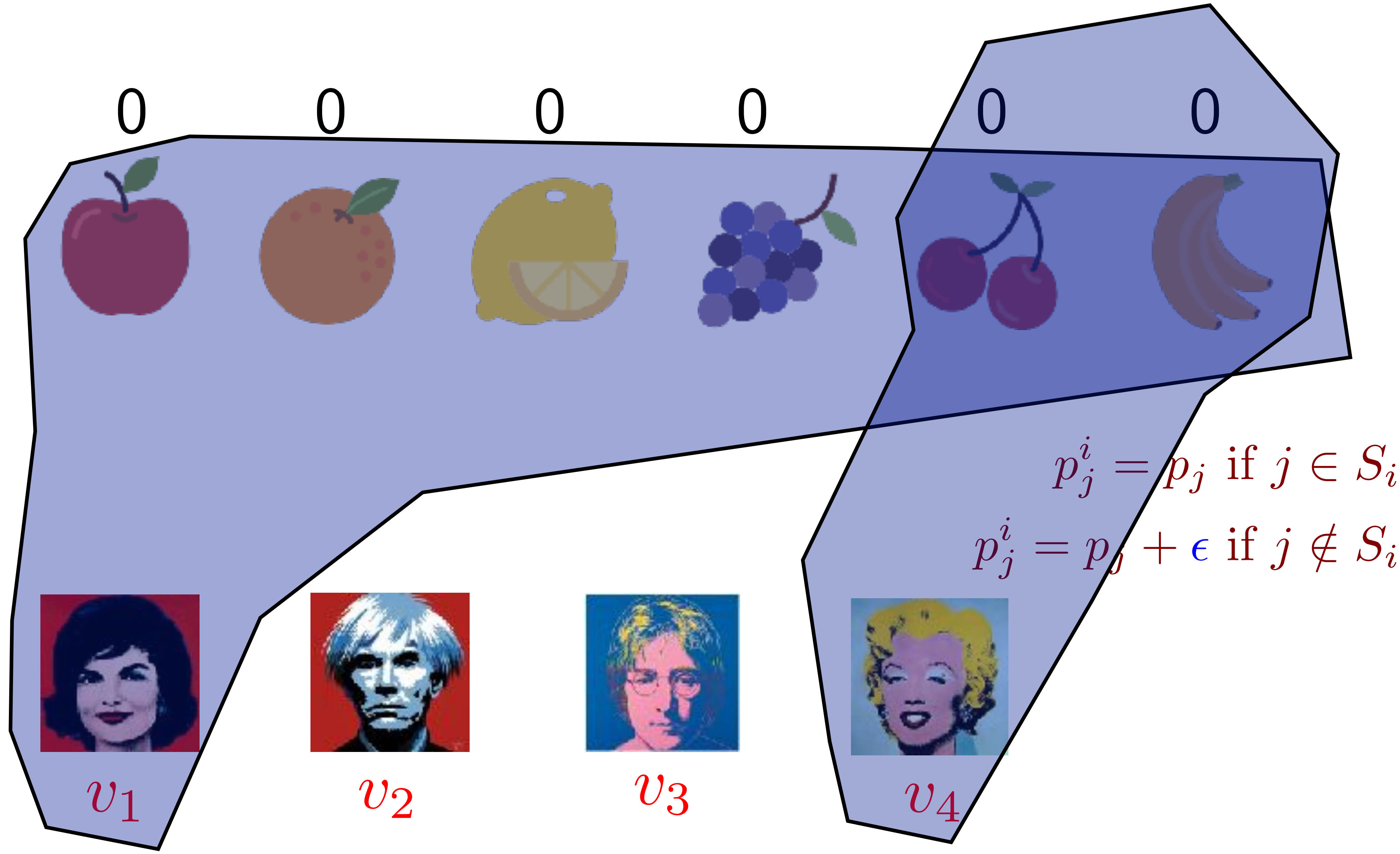

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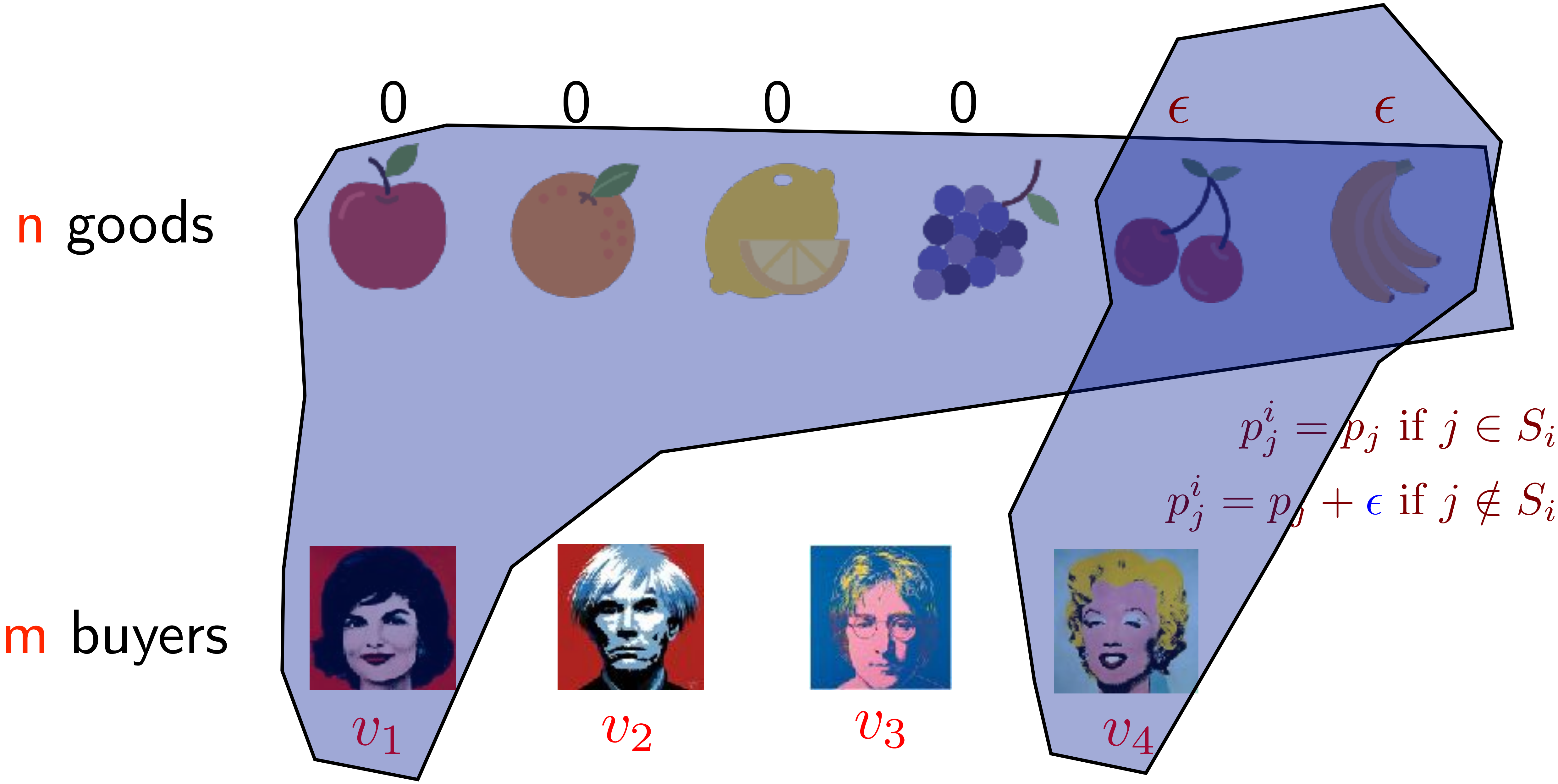

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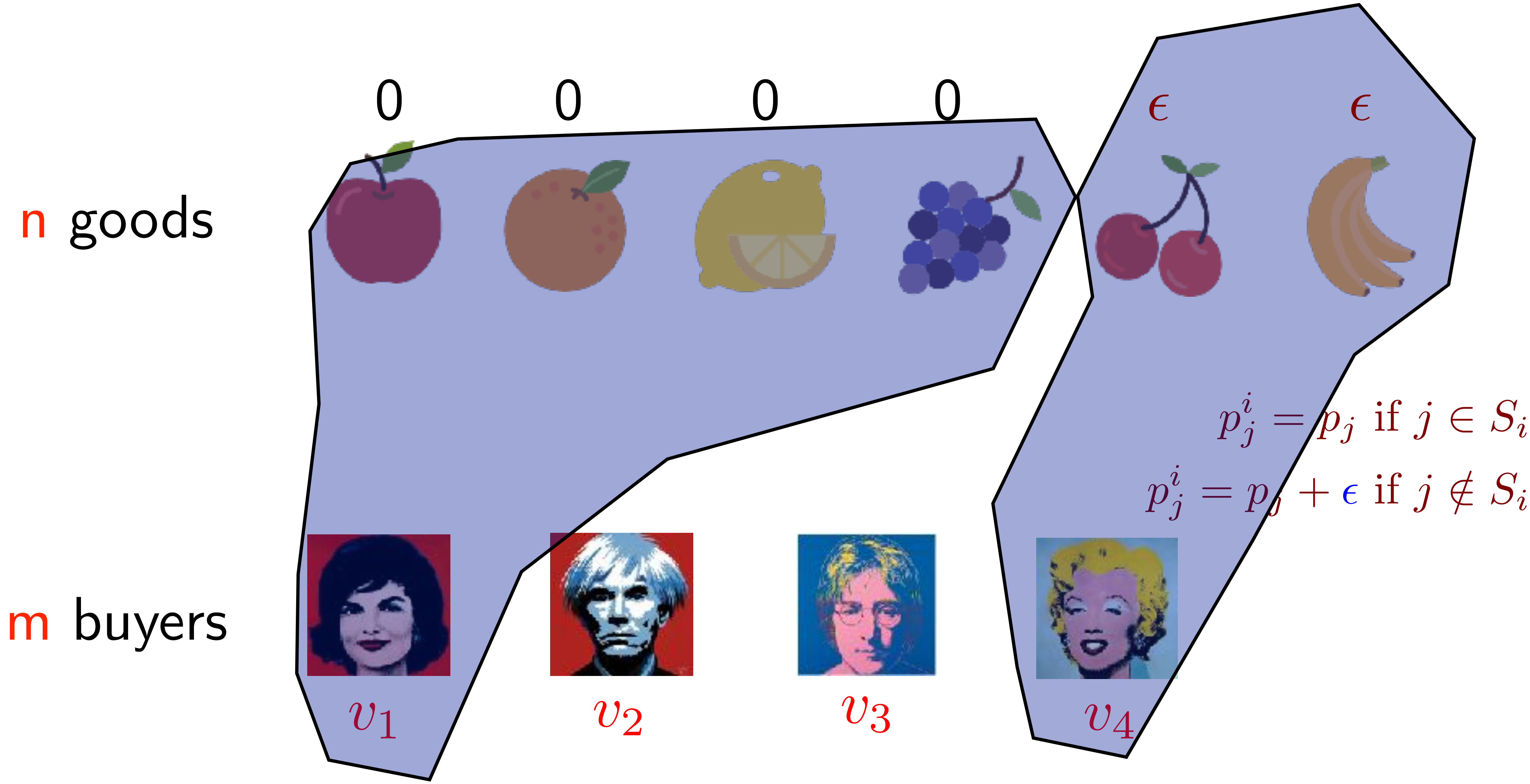

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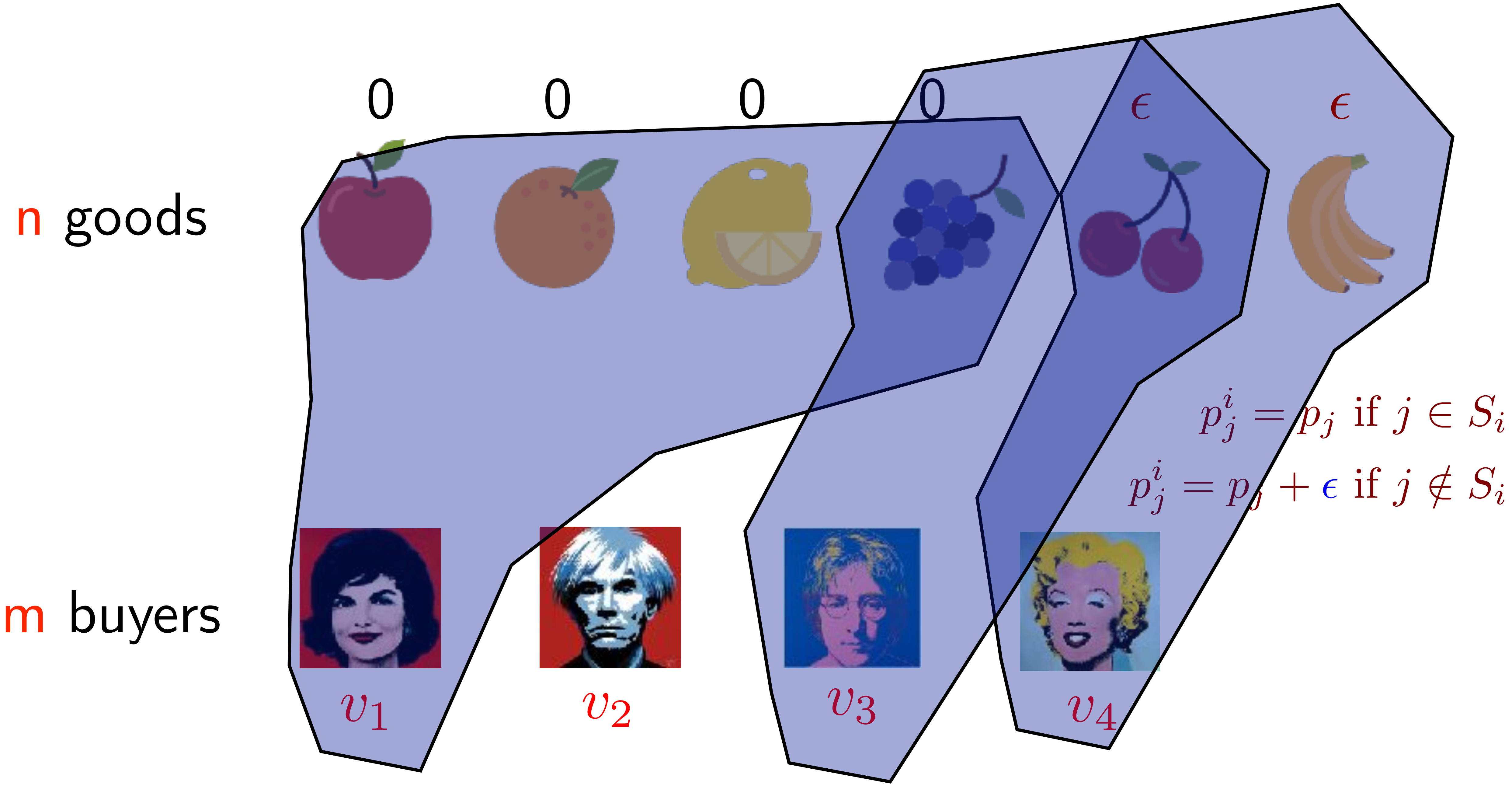

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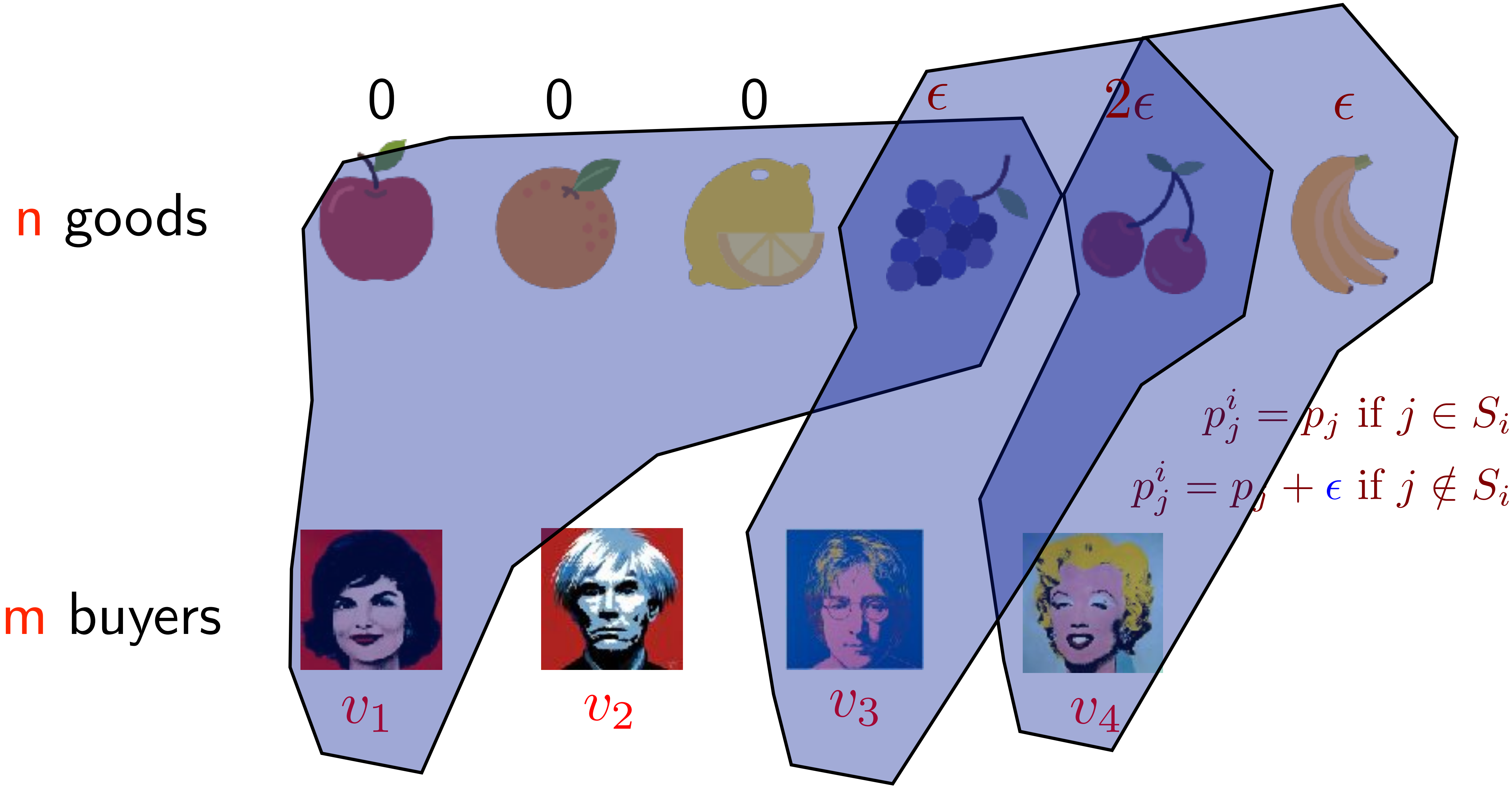

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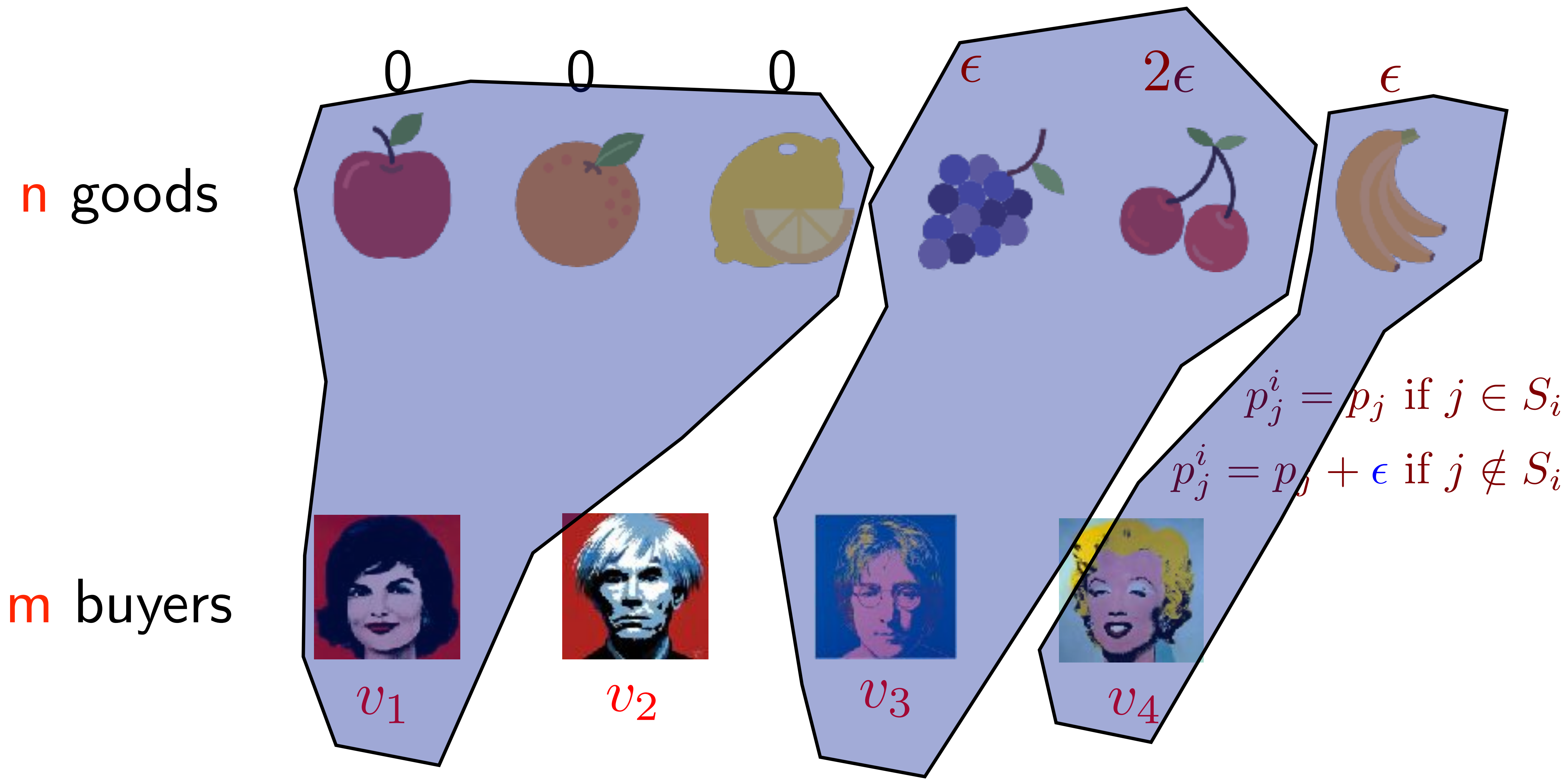

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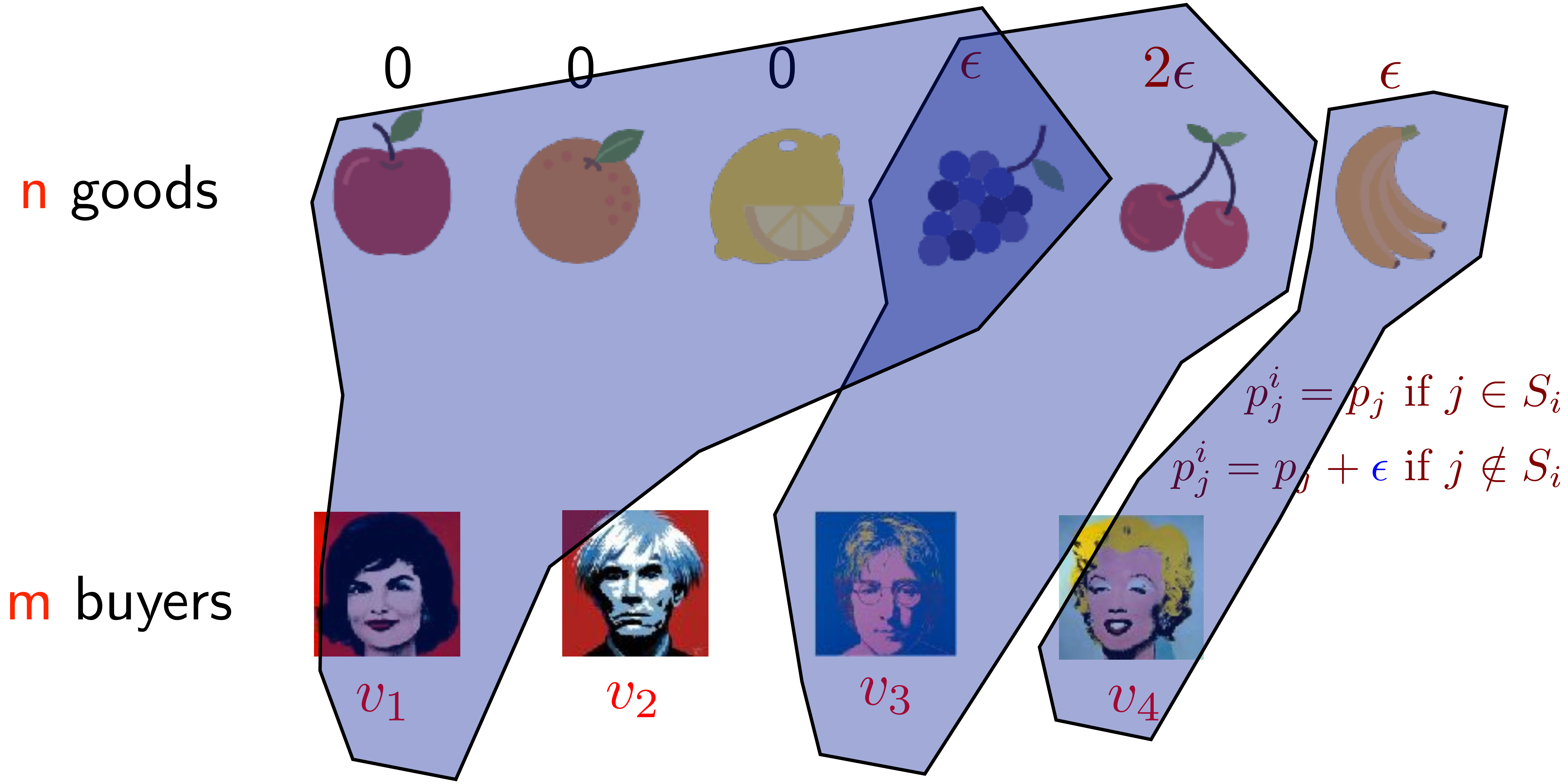

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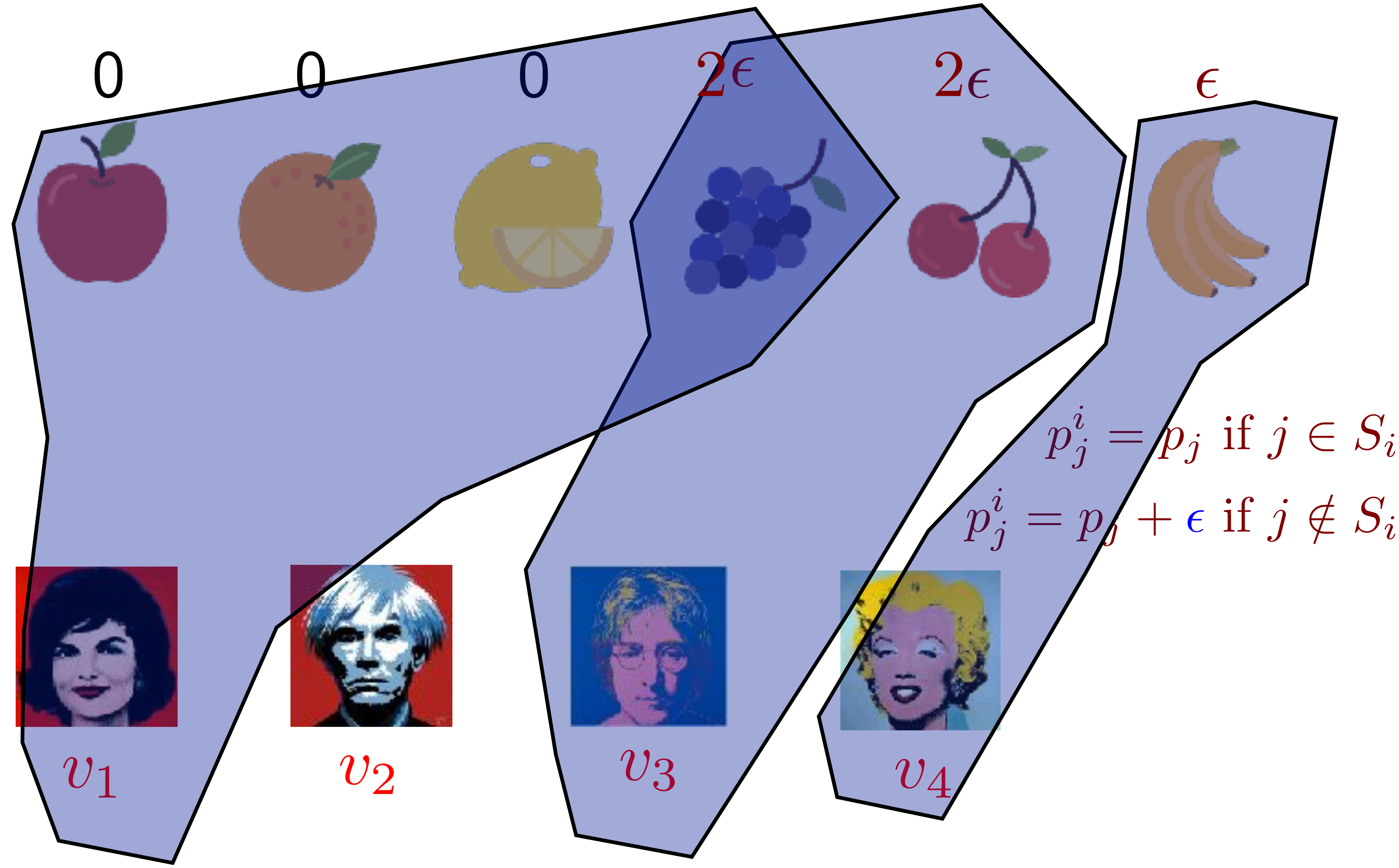

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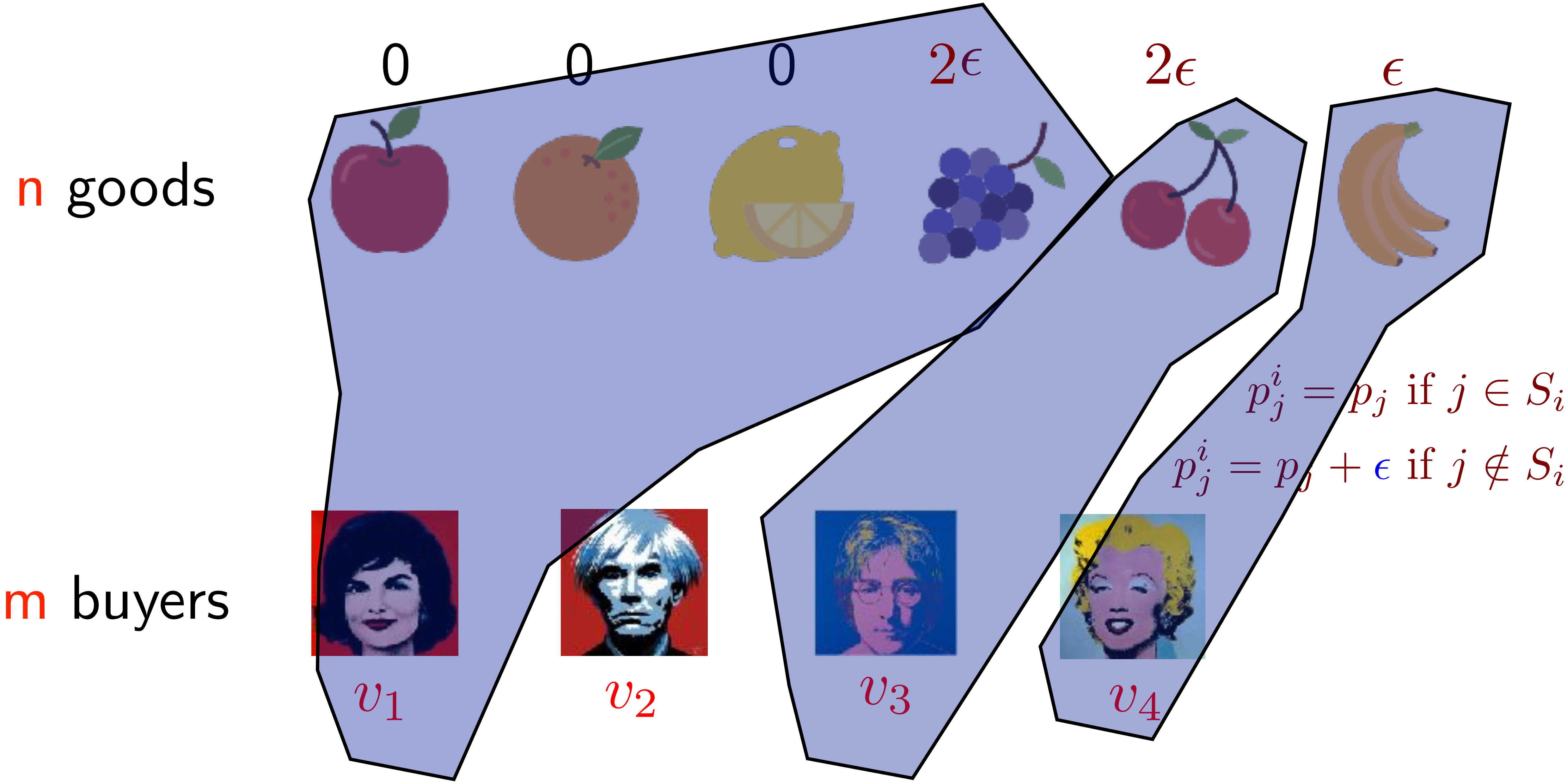

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- With the new definition, the algorithm always keeps a partition.

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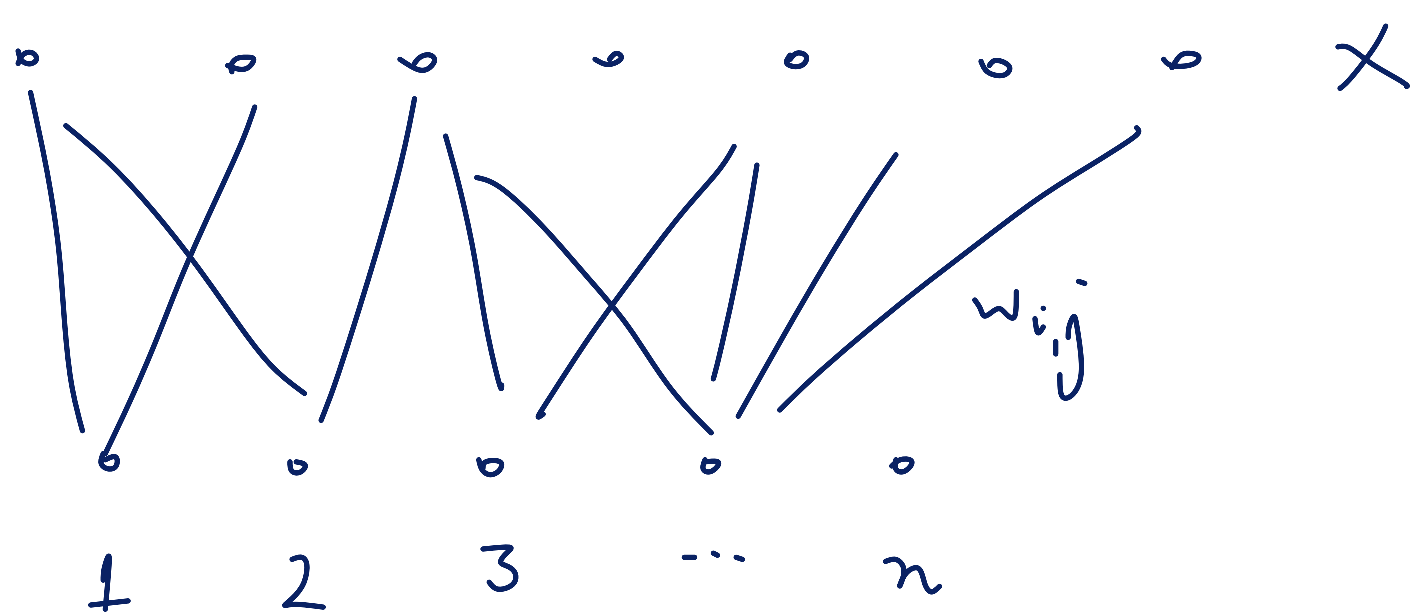

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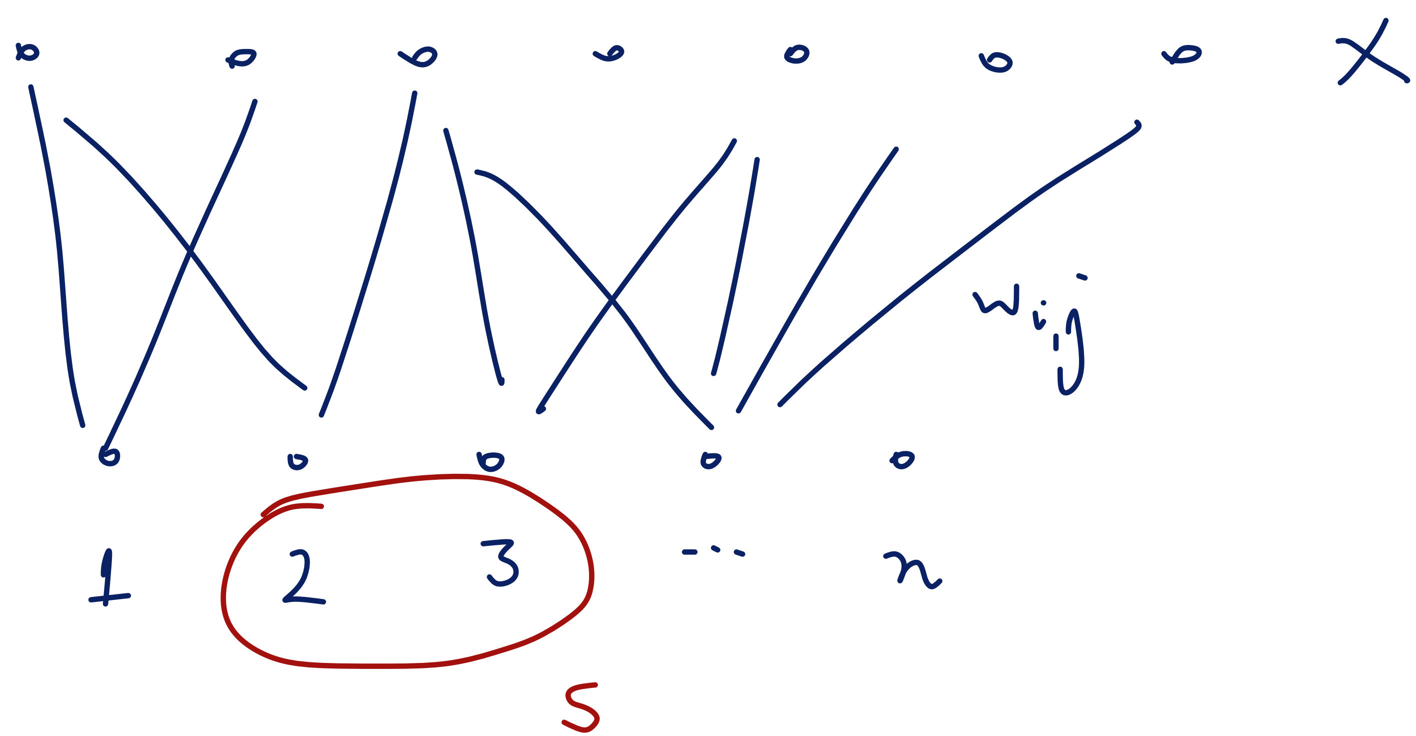

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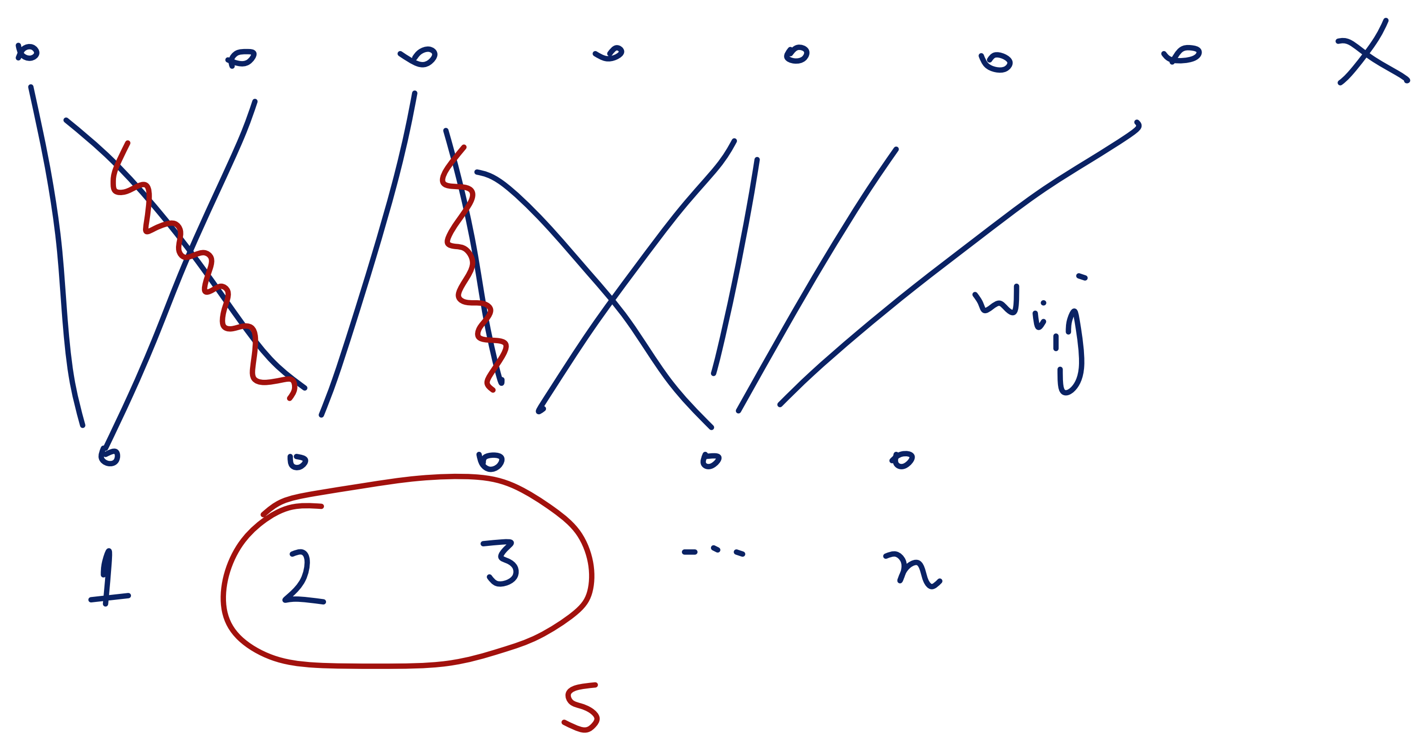

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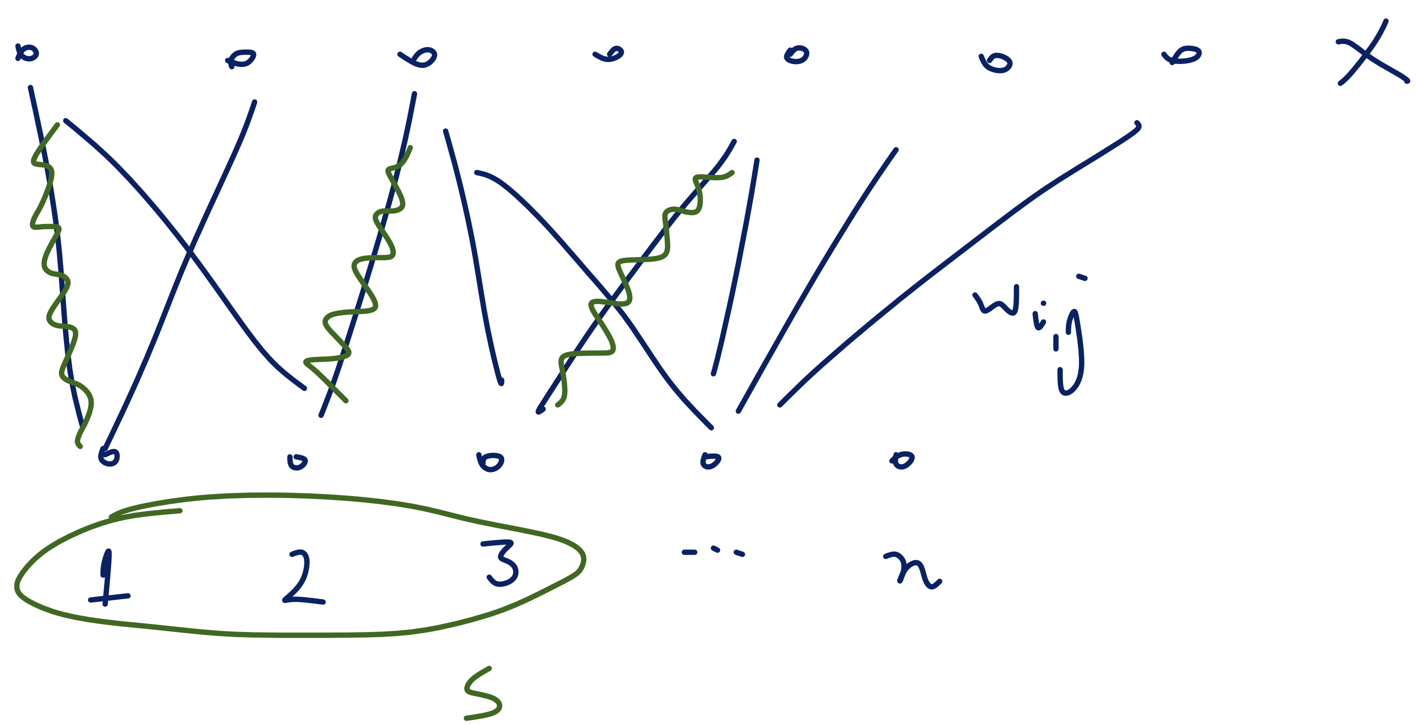

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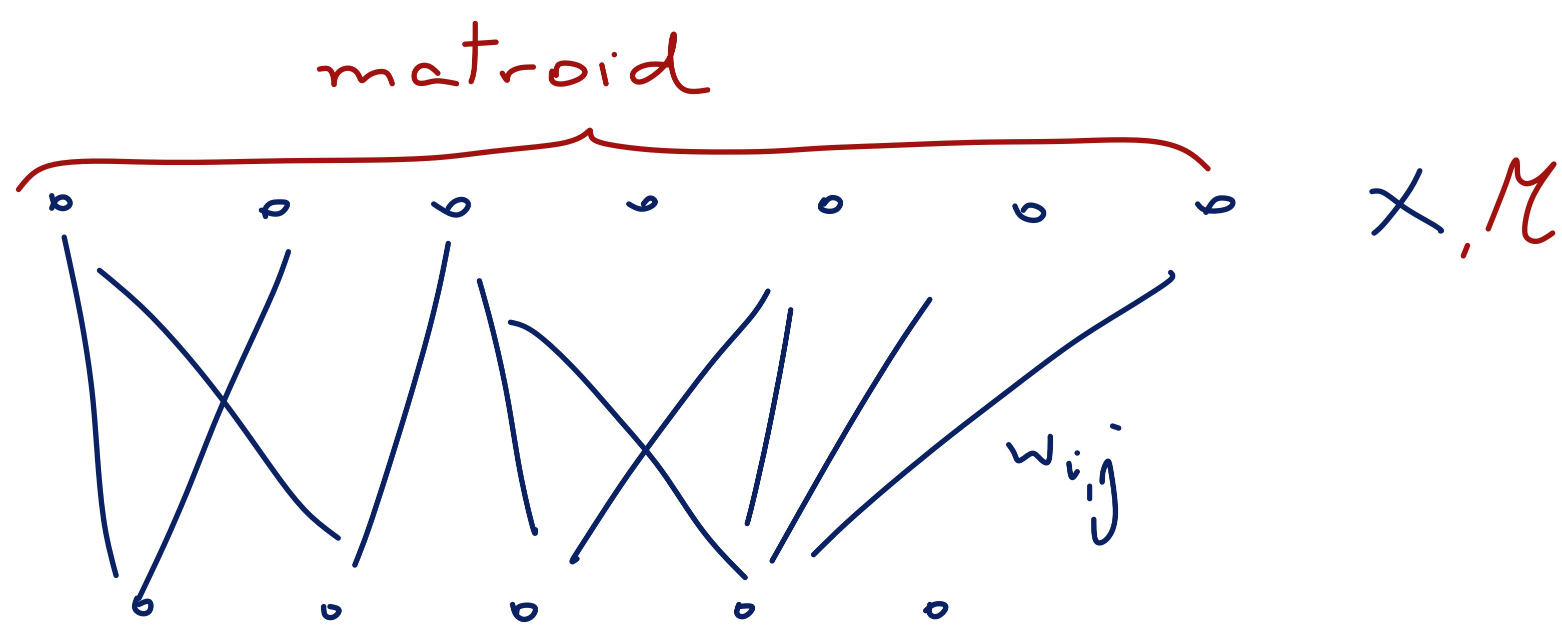

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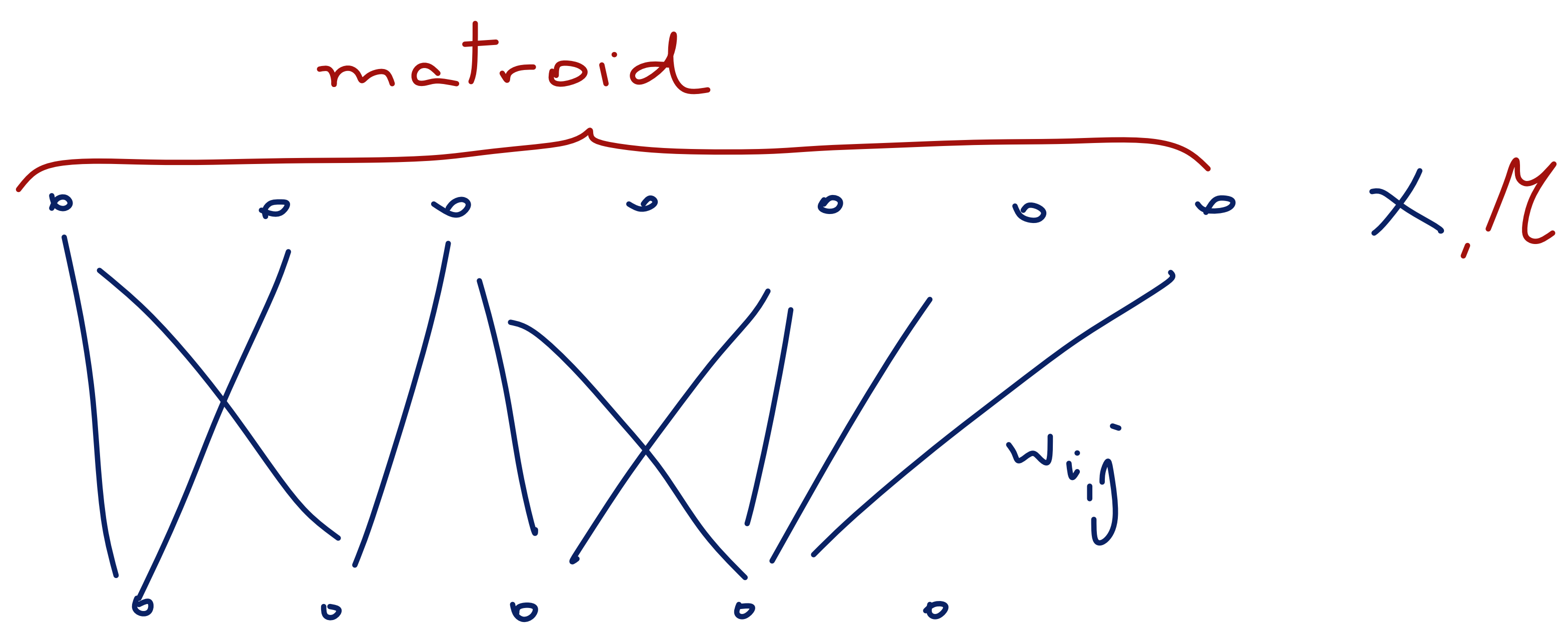

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- Theorem [Kelso-Crawford]: If all agents have GS valuations, then Walrasian equilibrium always exists.

- Theorem [Gul-Stachetti]: If a class C of valuations contains all unit-demand valuations and Walrasian equilibrium always exists then $\mathrm{C} \subseteq \mathrm{GS}$

Valuated Matroids

- Given vectors $v_1, \ldots, v_m \in \mathbb{Q}^n$ define

$$\psi_p(v_1, \ldots, v_n) = n \text{ if } \det(v_1, \ldots, v_n) = p^{-n} \cdot a/b$$

for p prime $a, b, p \in \mathbb{Z}$.

- Question in algebra:

$$\min_{v_i \in V} \psi_p(v_1, \ldots, v_n) \text{ s.t. } \det(v_1, \ldots, v_n) \neq 0$$

- Solution is a greedy algorithm: start with any non-degenerate set and go over each items and replace it by the one that minimizes $\psi_p(v_1, \ldots, v_n)$.

- [DW]: Grassmann-Plucker relations look like matroid cond

Valuated Matroids

- Definition: a function $v : \binom{[n]}{k} \to \mathbb{R}$ is a valuated matroid if the "Greedy is optimal".

Matroidal maps

- Definition: a function $v : 2^{[n]} \to \mathbb{R}$ is a matroidal map if for every $p \in \mathbb{R}^n$ a set in $D(v;p)$ can be obtained by the greedy algorithm : $S_0 = \emptyset$ and

$$S_t = S_{t-1} \cup \{i_t\} \text{ for } i_t \in \operatorname{argmax}_i v_p(i|S_t)$$

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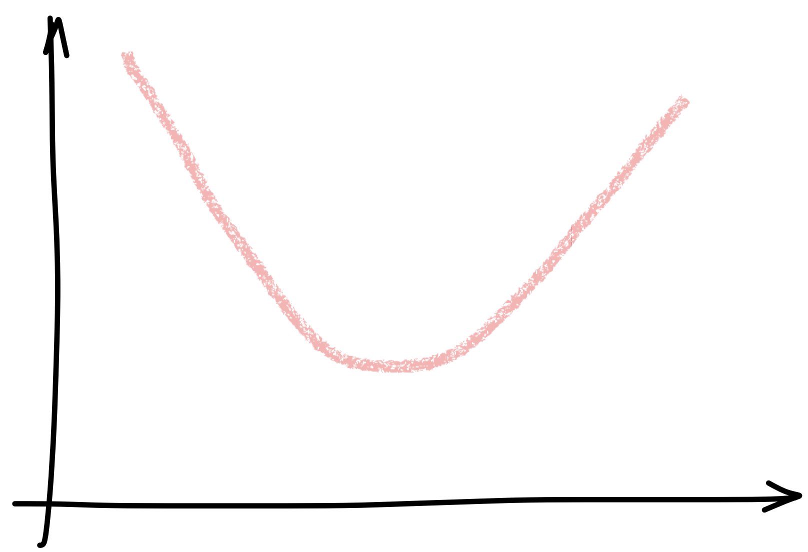
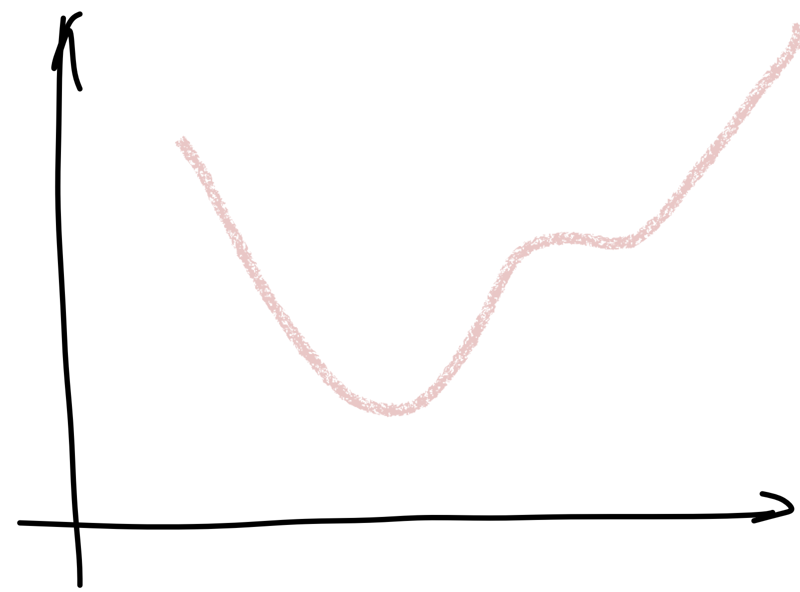
- Definition: a subset system $\mathcal{M} \subseteq 2^{[n]}$ is a matroid if for every $p \in \mathbb{R}^n$ the problem $\max_{S \in \mathcal{M}} p(S)$ can be solved by the greedy algorithm.

Discrete Concavity

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $p \in \mathbb{R}^n$, a local minimum of $f_p(x) = f(x) - \langle p, x \rangle$ is a global minimum.





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- We want to extend this notion to function in the hypercube: $v : 2^{[n]} \to \mathbb{R}$ (or lattice $v : \mathbb{Z}^{[n]} \to \mathbb{R}$ or other discrete sets such as the basis of a matroid)

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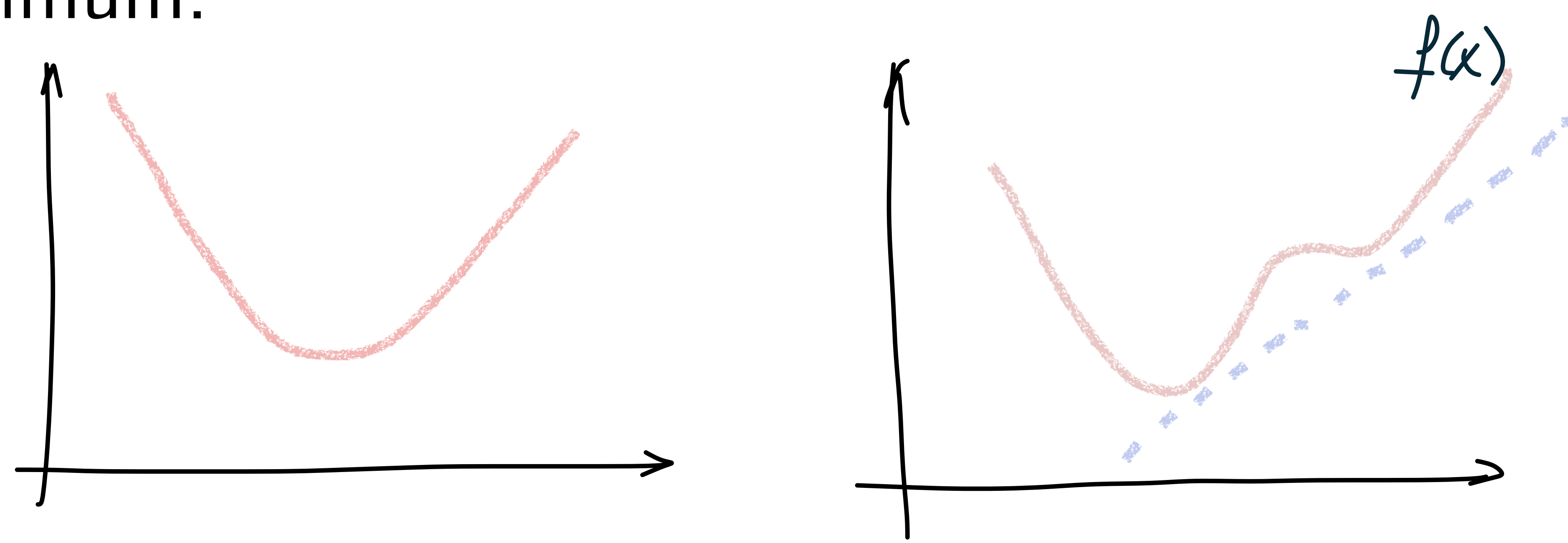

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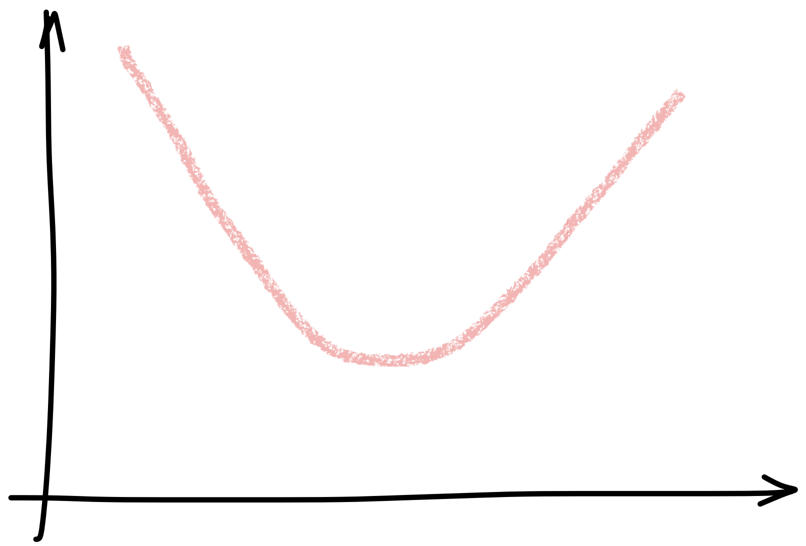

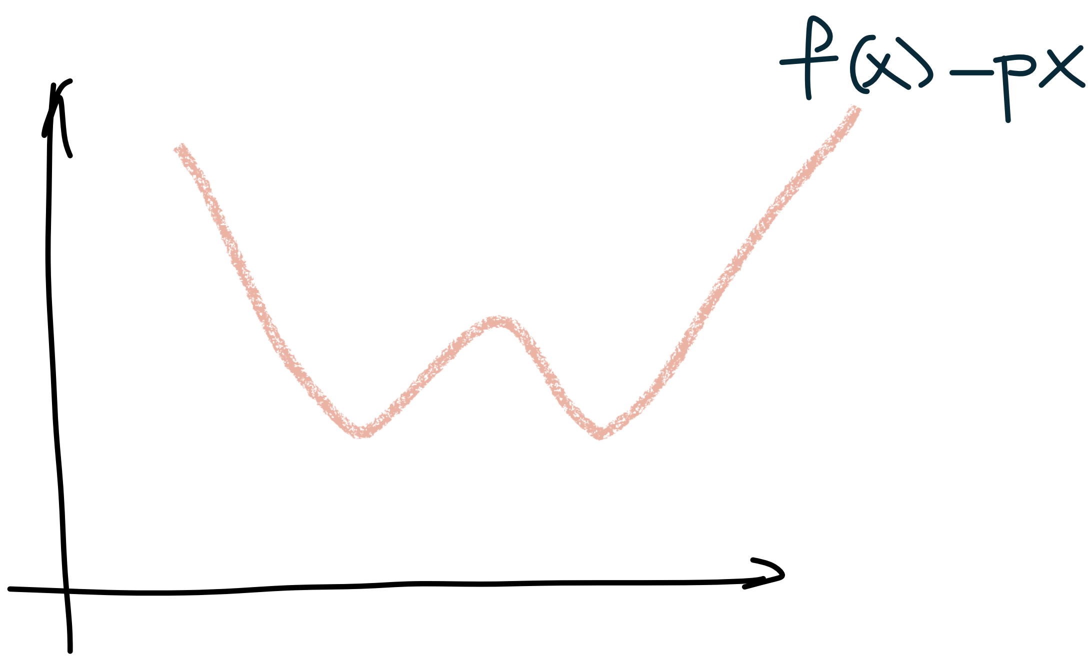

- Also, gradient descent converges for convex functions.
- We want to extend this notion to function in the hypercube: $v : 2^{[n]} \to \mathbb{R}$ (or lattice $v : \mathbb{Z}^{[n]} \to \mathbb{R}$ or other discrete sets such as the basis of a matroid)

Discrete Concavity

- A function $v : 2^{[n]} \to \mathbb{R}$ is discrete concave if for all $p \in \mathbb{R}^n$all local minima of v_p are global minima. I.e.

$$v_p(S) \geq v_p(S \cup i), \forall i \notin S$$
$$v_p(S) \geq v_p(S \setminus j), \forall j \in S$$
$$v_p(S) \geq v_p(S \cup i \setminus j), \forall i \notin S, j \in S$$

  then $v_p(S) \geq v_p(T), \forall T \subseteq [n]$. In particular local search always converges.

- [Murota '96] M-concave (generalize valuated matroids) [Murota-Shioura '99] $M^\natural$-concave functions

Equivalence

- [Fujishige-Yang] A function $v : 2^{[n]} \to \mathbb{R}$ is gross substitutes iff it is a matroidal map iff it is discrete concave.

[Kelso-Crawford '82]
necessary /"sufficient" condition for price adjustment to converge

gross substitutes

[Murota-Shioura '99]
generalize convexity to discrete domains

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- In particular $S \in D(v; p)$ in poly-time.
- Proof through discrete differential equations

Discrete Differential Equations

- Given a function $v : 2^{[n]} \to \mathbb{R}$ we define the discrete derivative with respect to $i \in [n]$ as the function $\partial_i v : 2^{[n] \setminus i} \to \mathbb{R}$ which is given by:

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- If we apply it twice we get:

$$\partial_{ij} v(S) := \partial_j \partial_i v(S) = v(S \cup ij) - v(S \cup i) - v(S \cup j) + v(S)$$

- Submodularity: $\partial_{ij} v(S) \leq 0$

Discrete Differential Equations

- [Reijnierse, Gellekom, Potters] A function $v : 2^{[n]} \to \mathbb{R}$ is in gross substitutes iff it satisfies:

$$\partial_{ij} v(S) \leq \max(\partial_{ik} v(S), \partial_{kj} v(S)) \leq 0$$

condition on the discrete Hessian.

- Idea: A function is in GS iff there is not price such that:

$D(v;p) = \{S, S \cup ij\}$ or $D(v;p) = \{S \cup k, S \cup ij\}$

If v is not submodular, we can construct a price of the first type. If $\partial_{ij} v(S) > \max(\partial_{ik} v(S), \partial_{kj} v(S))$ then we can find a certificate of the second type.

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Algorithmic Problems

- Techniques:
  - Tatonnement
  - Linear Programming
  - Gradient Descent
  - Cutting Plane Methods
  - Combinatorial Algorithms

Linear Programming

- [Nisan-Segal] Formulate this problem as an LP:

$$\max \sum_i v_i(S) x_{iS}$$

$$\sum_S x_{iS} = 1, \forall i \in [m]$$

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- For GS, the IP is integral: $W_{\text{IP}} \leq W_{\text{LP}} = W_{\text{D-LP}}$
- Consider a Walrasian equilibrium and p the Walrasian prices and u the agent utilities. Then it is a solution to the dual, so: $W_{\text{D-LP}} \leq W_{\text{eq}} = W_{\text{IP}}$

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- In general, Walrasian equilibrium exists iff LP is integral.

- Separation oracle for the dual: $u_i \geq \max_S v_i(S) - p(S)$ is the demand oracle problem.

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dual

- Walrasian equilibrium exists + demand oracle in poly-time = Welfare problem in poly-time
- [Roughgarden, Talgam-Cohen] Use complexity theory to show non-existence of equilibrium, e.g. budget additive.

Gradient Descent

- We can Lagrangify the dual constraints and obtain the following convex potential function:

$$\phi(p) = \sum_i \max_S [v_i(S) - p(S)] + \sum_j p_j$$

- Theorem: the set of Walrasian prices (when they exist) are the set of minimizers of ϕ.

$$\partial_j \phi(p) = 1 - \sum_i 1[j \in S_i]; S_i \in D(v_i; p)$$

- Gradient descent: increase price of over-demanded items and decrease price of over-demanded items.
- Tatonnement: $p_j \leftarrow p_j - \epsilon \cdot \operatorname{sgn} \partial_j \phi(p)$

Comparing Methods

method	oracle	running-time

How to access the input

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Value oracle:

given i and S:

query $v_i(S)$.

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Aggregate Demand:
given p, query.
$\sum_i S_i; S_i \in D(v_i, p)$

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method	oracle	running-time
tatonnement/GD	aggreg demand	pseudo-poly

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Comparing Methods

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tatonnement/GD	aggreg demand	pseudo-poly
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cutting plane	aggreg demand	weakly-poly

- [PL-Wong]: We can compute an exact equilibrium with $\tilde{O}(n)$ calls to an aggregate demand oracle.

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- [Murota]: We can compute an exact equilibrium for gross susbtitutes in $\tilde{O}((mn + n^3)T_V)$ time.

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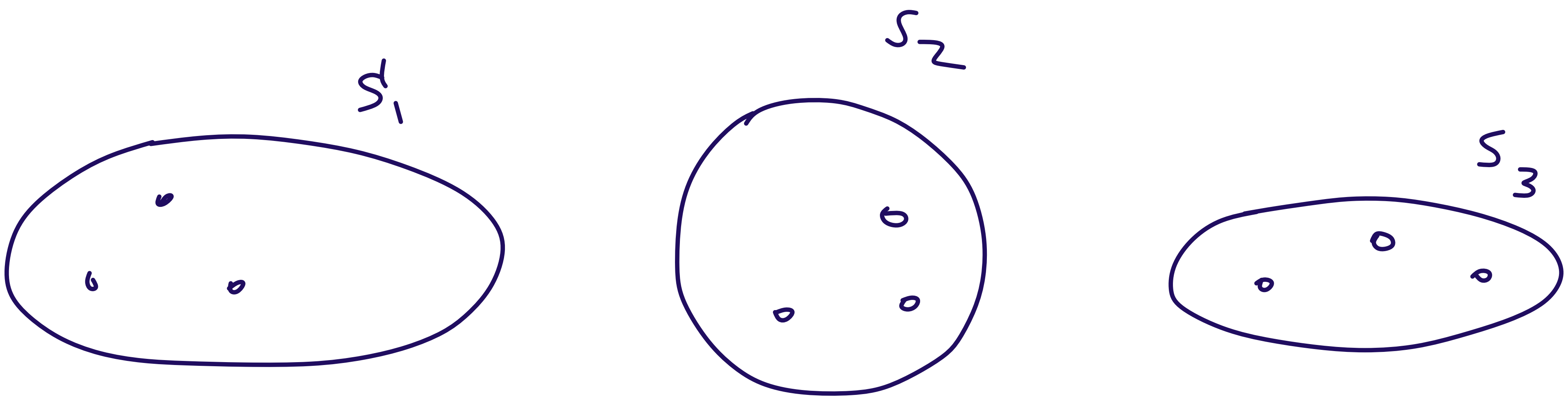
Computing Walrasian prices

- Given a partition $S_1, \ldots, S_m$ we want to find prices such that $S_i \in \text{argmax}_S\, v_i(S) - p(S)$
- For GS, we only need to check that no buyer want to add, remove or swap items.



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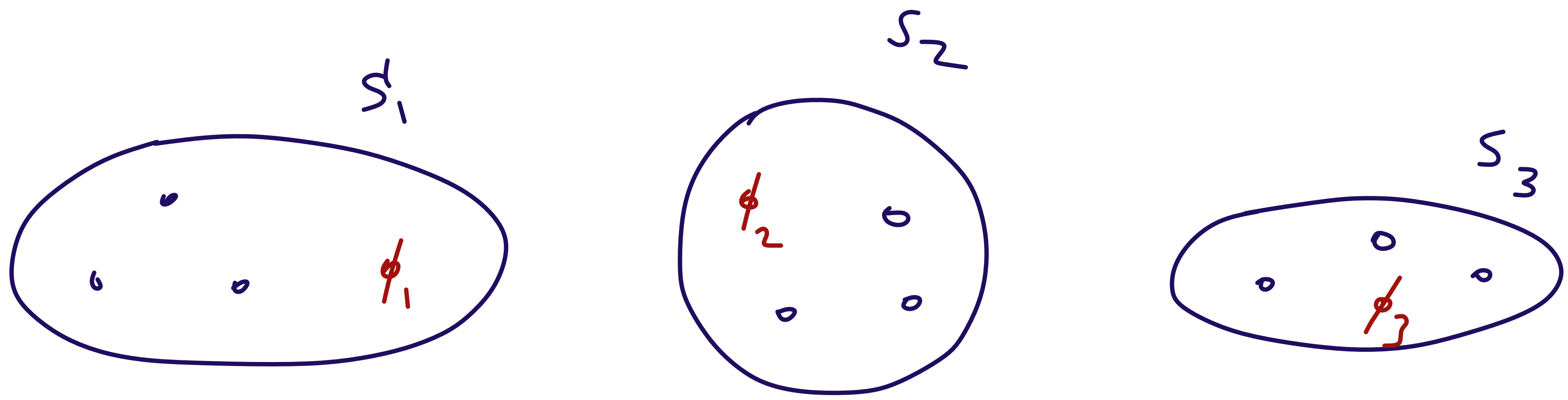

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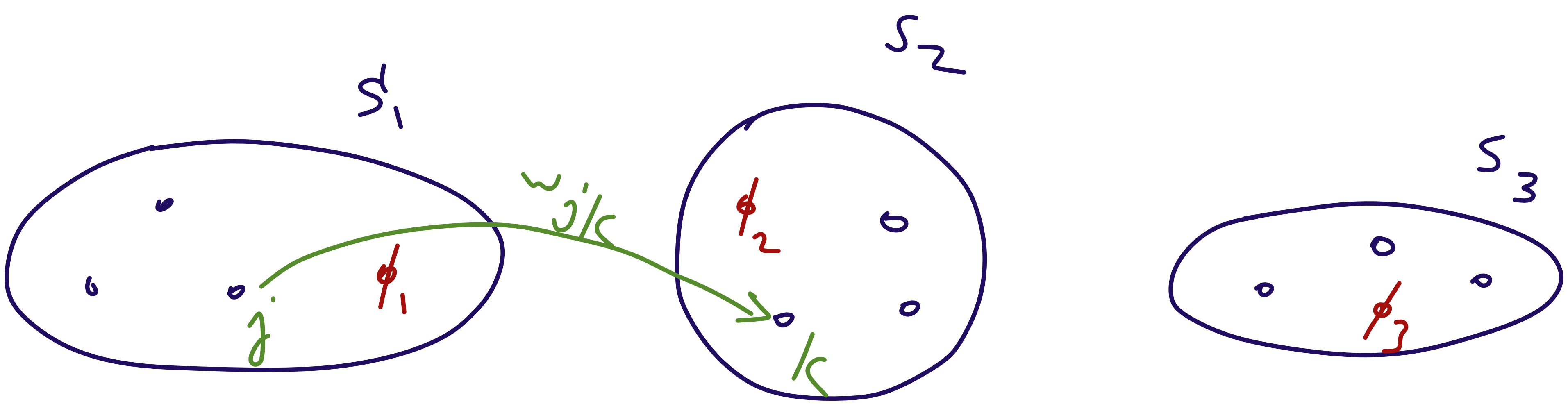

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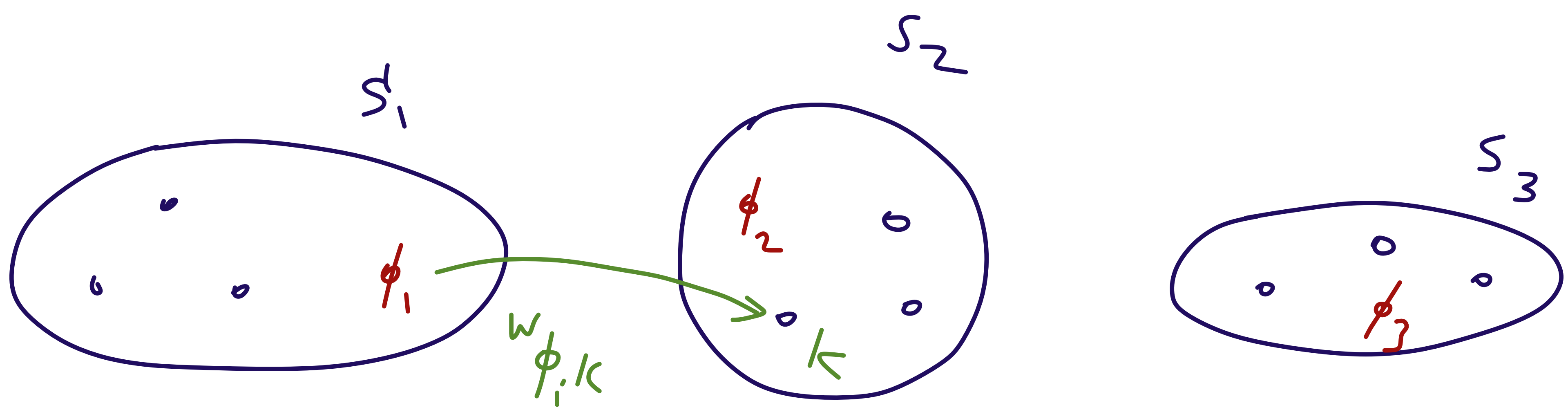

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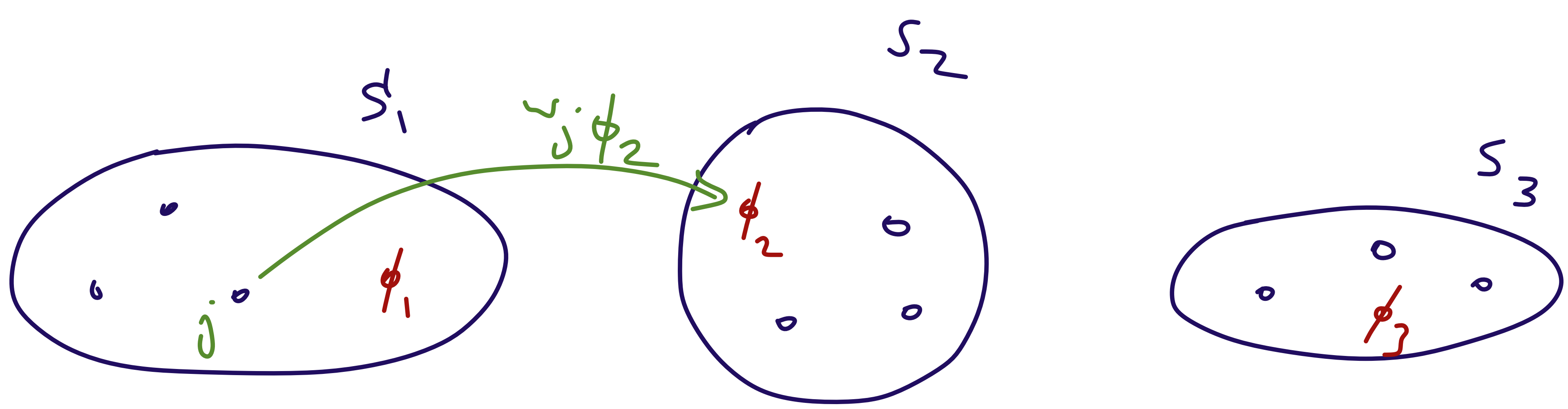

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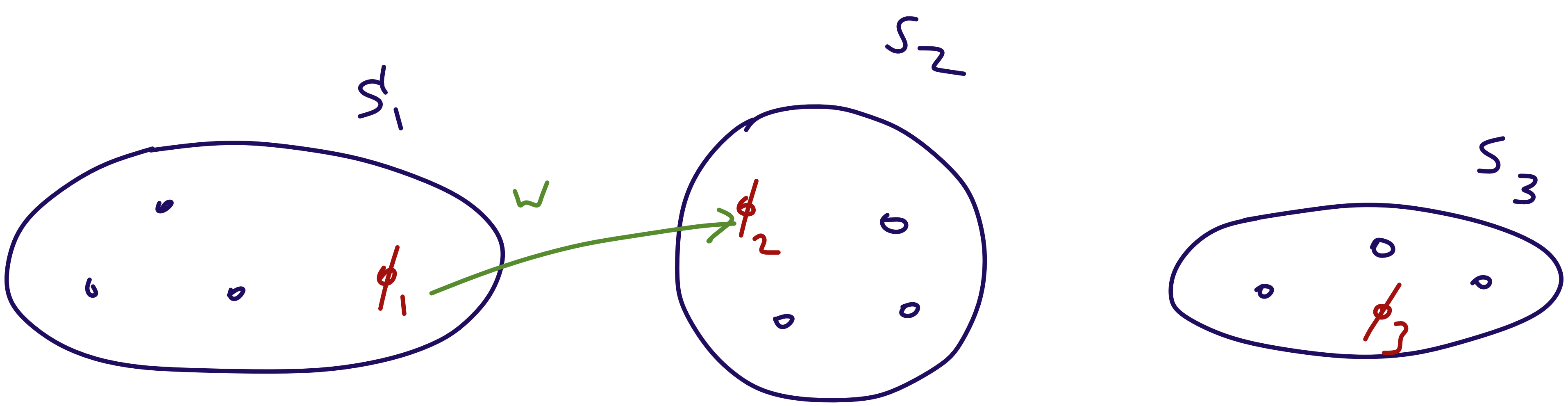

$w_{\phi_i \phi_{i'}} = 0$

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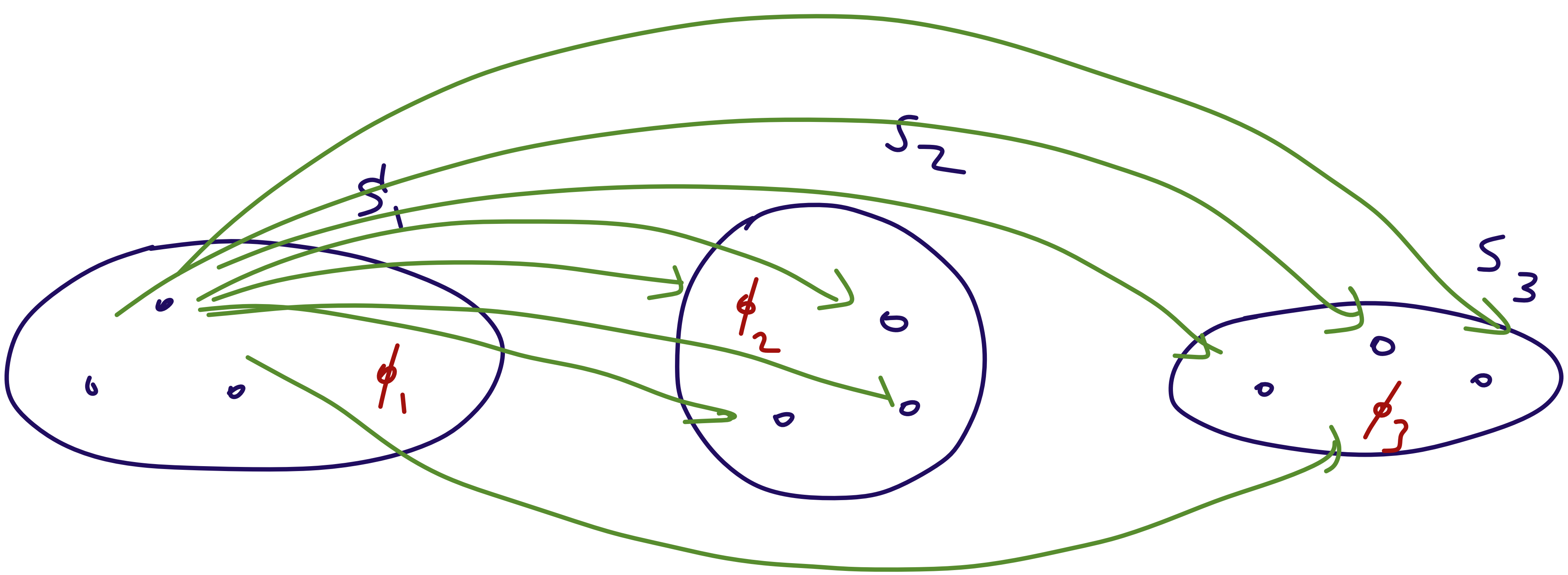

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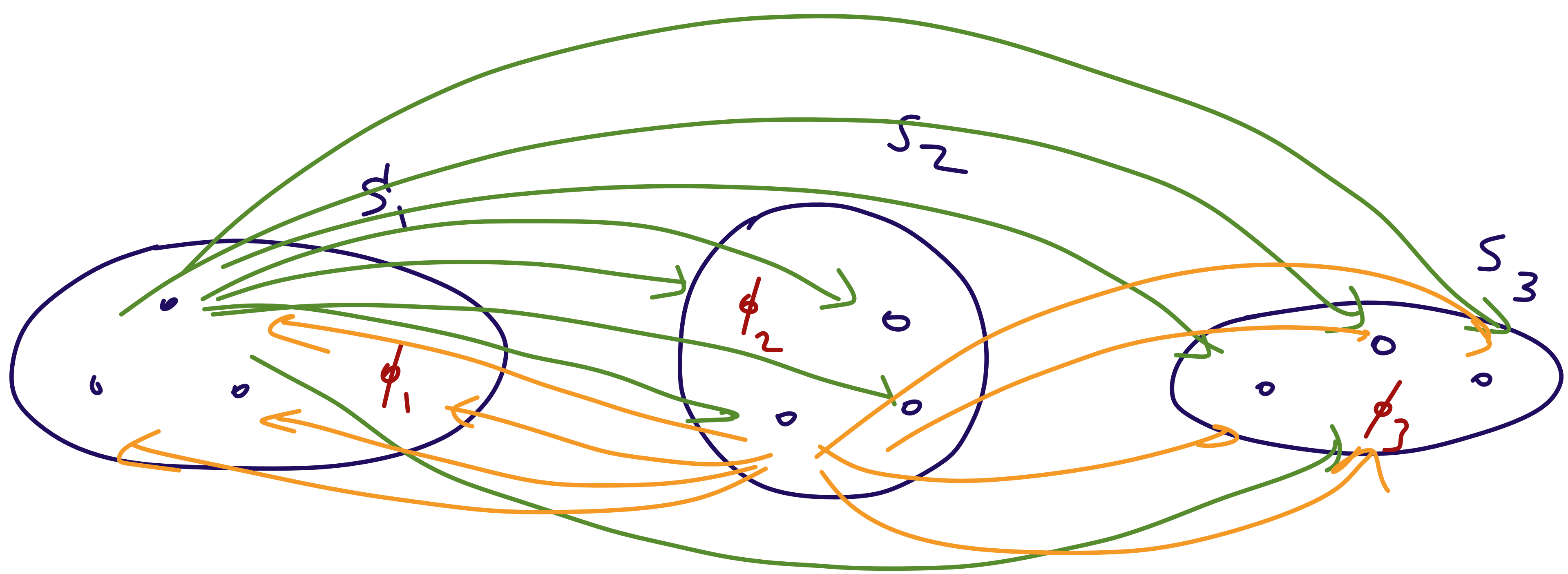

Computing Walrasian prices

- Theorem: the allocation is optimal if the exchange graph has no negative cycle.
- Proof: if no negative cycles the distance is well defined. So let $p_j = -\mathrm{dist}(\phi, j)$ then:

$$\mathrm{dist}(\phi, k) \leq \mathrm{dist}(\phi, j) + w_{jk}$$

$$v_i(S_i) \geq v_i(S_i \cup k \setminus j) - p_k + p_j$$

And since S_i is locally-opt then it is globally opt.
Conversely: Walrasian prices are a dual certificate showing that no negative cycles exist.

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- Nice consequence: Walrasian prices form a lattice.

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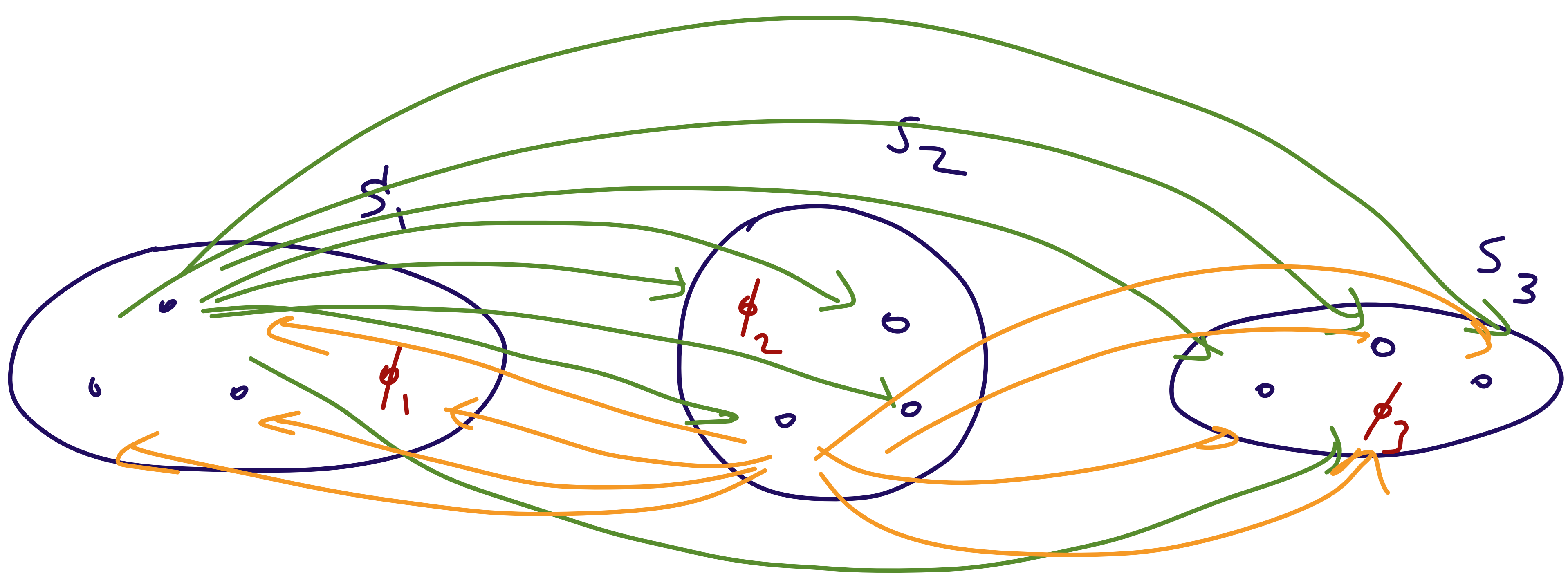
Incremental Algorithm

- For each $t = 1..n$ we will solve problem W_t to find the optimal allocation of items $[t] = \{1..t\}$ to m buyers.
- Problem W_1 is easy.
- Assume now we solved W_t getting allocation $S_1, \ldots, S_m$ and a certificate p = maximal Walrasian prices.



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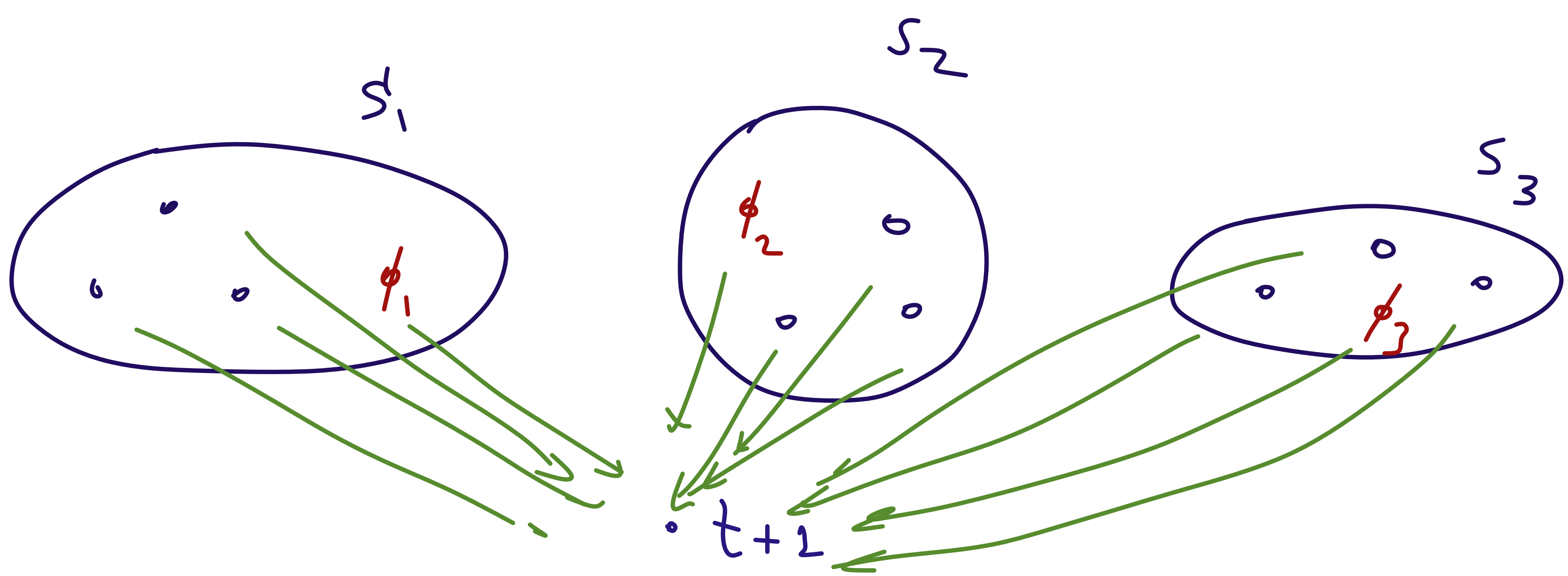

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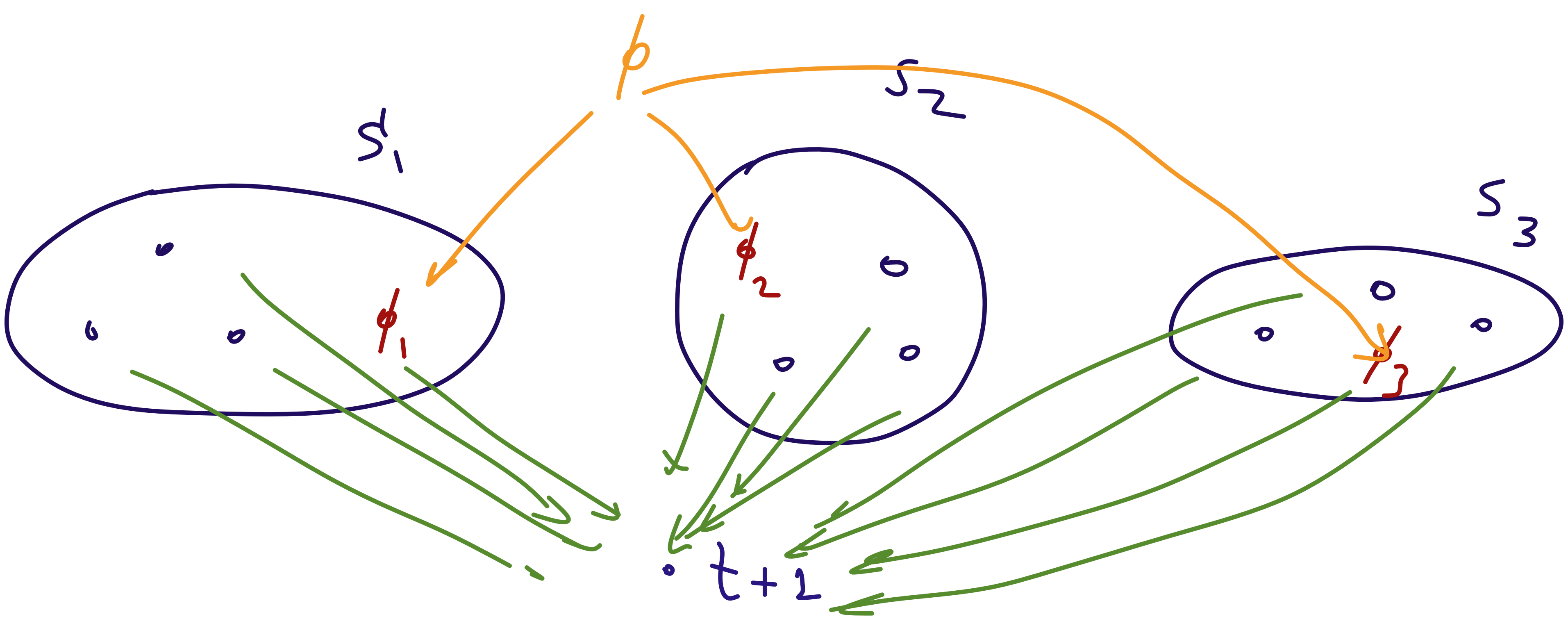

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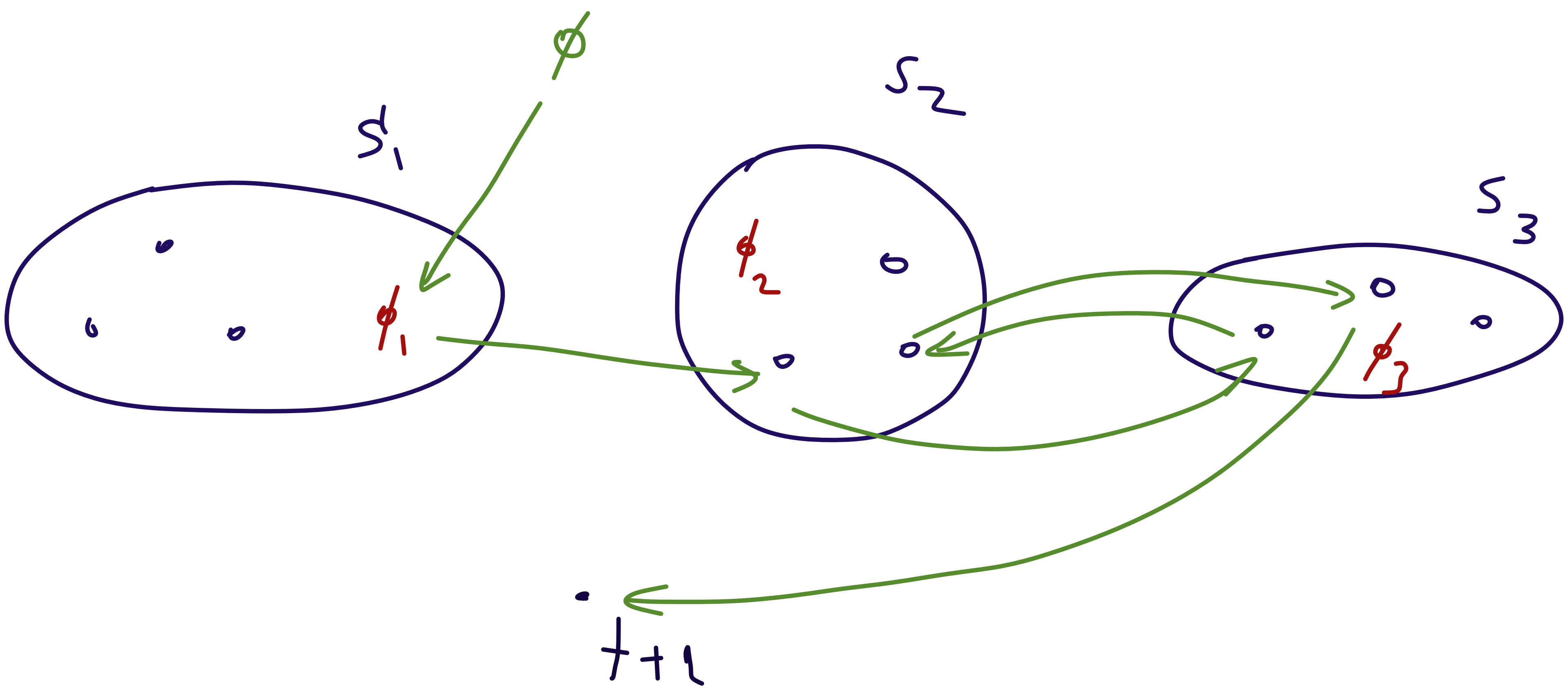

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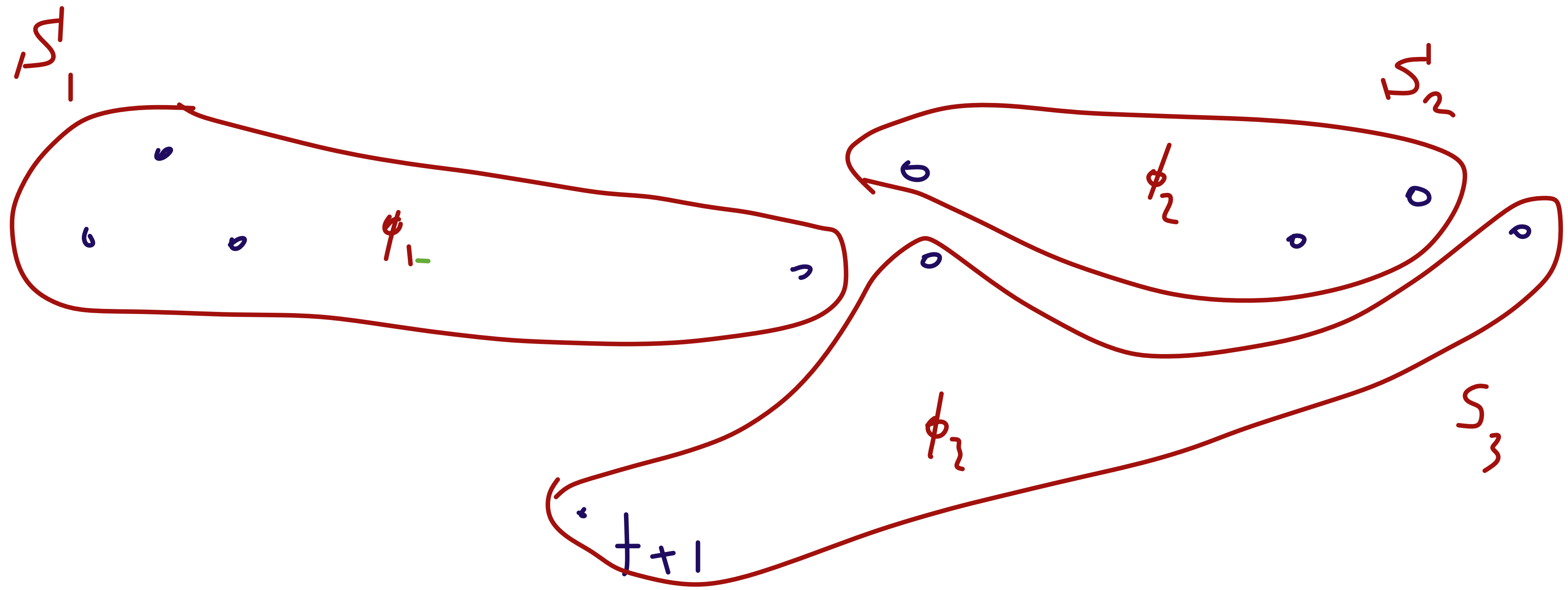

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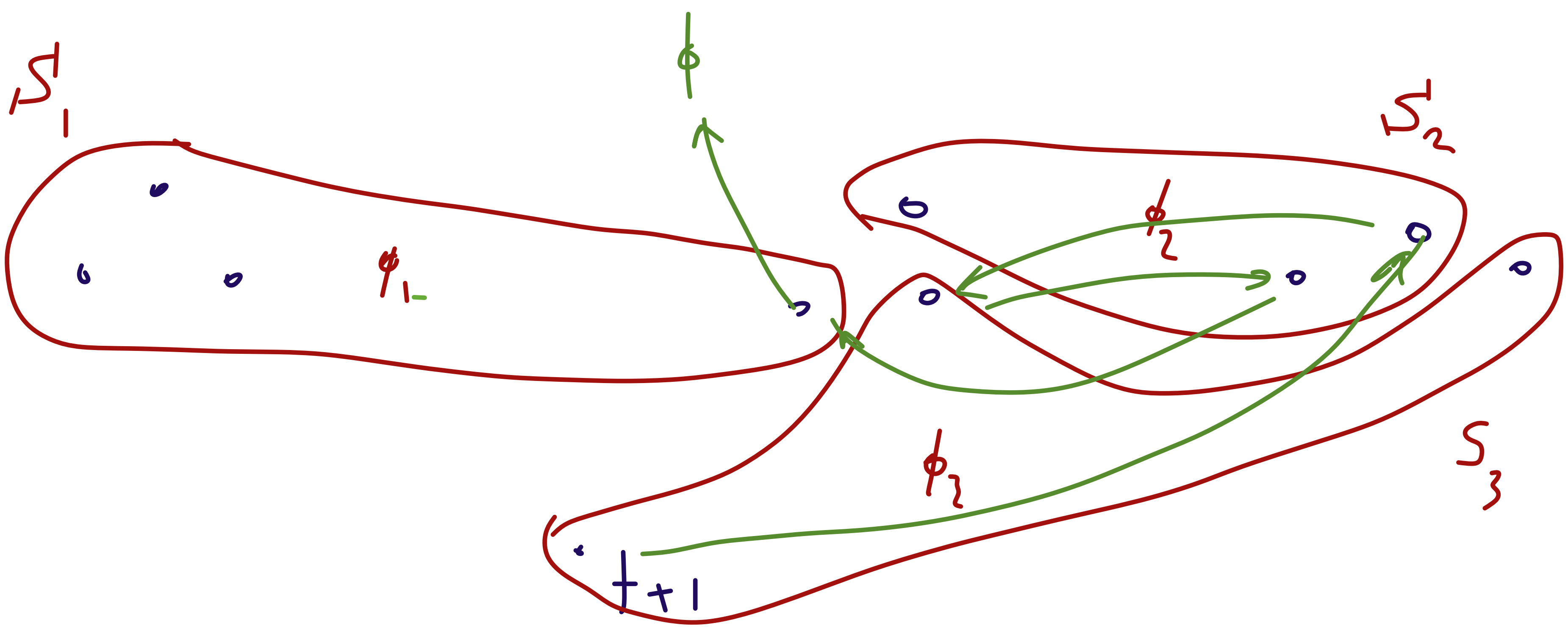

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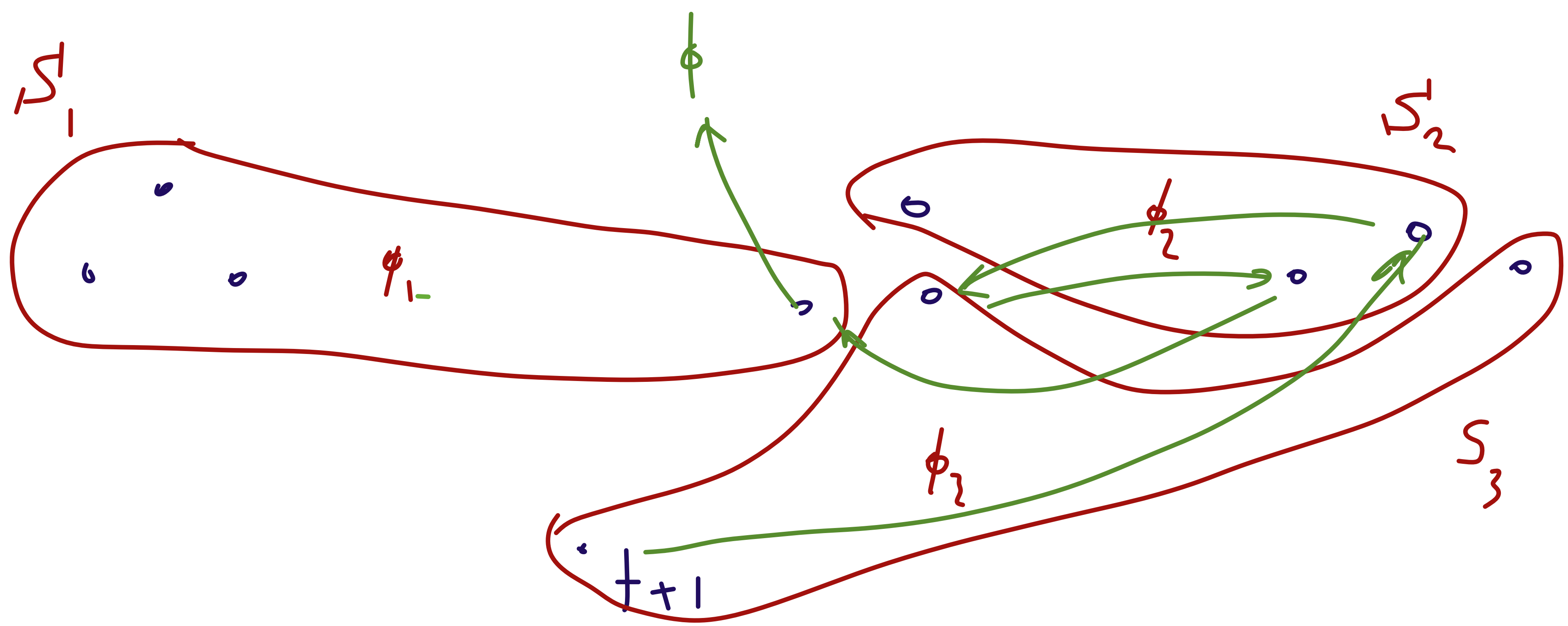

- Graph has $O(t^2 + mt)$ non-negative edges
- After n iterations of Dijkstra we get $\tilde{O}(n^3 + n^2 m)$

Incremental Algorithm

- Proof that new allocation $\tilde{S}_1 \ldots \tilde{S}_m$ is optimal
- Define the new prices $\tilde{p}_j = -\operatorname{dist}(\phi, j)$
  - (1) New prices are also a certificate for $S_1 \ldots S_m$
  - (2) $v_i(S_i) - \tilde{p}(S_i) = v_i(\tilde{S}_i) - \tilde{p}(\tilde{S}_i)$
  - Hence, $\tilde{S}_1 \ldots \tilde{S}_m$ and $\tilde{p}$ are Walrasian prices.

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  - endowment $\tilde{v}(S) = v(S|X)$
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- Open question: can we construct all gross substitutes from matroid rank functions and those operations ?
  - Some progress: See talk by Eric Balkanski on Thu

End of Part I