Sponsored Search Equilibria for Conservative Bidders

Renato Paes Leme
Department of Computer Science
Cornell University
Ithaca, NY
renatoppl@cs.cornell.edu

Éva Tardos *
Department of Computer Science
Cornell University
Ithaca, NY
eva@cs.cornell.edu

ABSTRACT
Generalized Second Price Auction and its variants has been the main mechanism used by search companies to auction positions for sponsored search links. In this paper we study the social welfare of the Nash equilibria of this game. It is known that socially optimal Nash equilibria exists, and its not hard to see that in the general case there are also very bad equilibria: the gap between a Nash equilibrium and the socially optimal can be arbitrarily large. In this paper, we consider the case when the bidders are conservative, in the sense that they do not bid above their own valuations. We show that a certain analog of the trembling hand equilibria are equilibria with conservative bidders. Our main result is to show that for conservative bidders the worse Nash equilibrium and the social optimum are within a factor of the golden ratio, 1.618.

Keywords
Game Theory, Keyword Auctions

1. INTRODUCTION
Search engines and other online information sources use sponsored search auction to monetize their services. These actions allocate advertisement slots to companies, and companies are charged pay per click, that is, they are charged a fee for any user that clicks on the link associated with the advertisement. The fee for such a click is decided by variant of the so-called **Generalized Second Price** Auction (GSP), a simple generalization of the well-known Vickrey auction [10] for a single item (or a single advertising slot). The Vickrey auction [10] for a single item, and its generalization, the Vickrey-Clarke-Groves Mechanism (VCG) [2, 5], make truthful behavior (when the advertisers reveal their true valuation) dominant strategy, and make the resulting outcome maximize the social welfare. See also [1] about truthful sponsored search auctions.

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Generalized Second Price Auction, the mechanism adopted by all search companies, is a natural generalization of the Vickrey auction for a single slot, but it is neither truthful nor maximizes social welfare. In this paper we will consider the social welfare of the GSP auction outcomes. Our goal in this paper is to show that the intuition based on the similarity of GSP to the truthful Vickrey auction is not so far from truth: we prove that the social welfare is within a factor of 1.618 of the optimal in any Nash equilibrium for conservative bidders.

We consider the full information game, assuming all advertisers know the valuations of all players. In addition, we will assume that the players are conservative, and do not risk bidding above their valuation. A bid value b_i above the valuation v_i for a player i, opens the player up to the risk of an outcome with negative utility (if another bidder b^* appears in the range $v_i < b^* < b_i$). To formally justify our conservative bidder model, we assume that an additional random bid will show up with a small ϵ probability, and study the Nash equilibria of the game for the original bidders as ϵ tends to zero. This is analogous to the traditional notion of trembling hand equilibrium [8]. We'll show that in the Nash equilibria of the game that survive this perturbation all bidders are conservative.

Our results. Our focus in this work is to analyze the social welfare in the Generalized Second Price Auction mechanism. We start by considering the simple model when click-though rates depend only on the slots, i.e., the probability of click for all bidders if assigned to slot i is α_i. At the end of the paper, we extend our results to the model with separable click-through rate, where if advertiser j is assigned to slot i the probability of this resulting in a click is $\gamma_j \alpha_i$. It is known that there are Nash equilibria that are socially optimal. We show simple examples of Nash equilibria where the social welfare is arbitrarily smaller than the optimum. However, these equilibria are unnatural, as some bid exceeds the players valuations, and hence the player takes unnecessary risk by playing above their own valuation if a new bidder shows up between their bid and valuation. We define conservative bidders as bidders who won't bid above their valuations.

Our main contribution is to prove that if all bidder are conservative, then the social welfare in a Nash equilibrium can't be very far from the optimal. To analyze the Nash equilibrium when all advertisers are conservative, we exhibit a simple property of those equilibria: consider two slots i and j, and let v_k denote the valuation of advertiser k for a click. We show that if in a Nash equilibrium with conservative bidders, $\pi(i)$ and $\pi(j)$ are assigned to these slots respectively, than we must have that

$$\frac{\alpha_j}{\alpha_i} + \frac{v_{\pi(i)}}{v_{\pi(j)}} \geq 1.$$

We say that as assignment of bidders to slots is weakly feasible if it satisfies the above inequality for all i and j, and we show that the social welfare of a weakly feasible assignment is at least a 1.618 fraction of the socially optimal assignment. Although only a necessary condition, weak feasibility is a simple and intuitive property. It is not hard to see that weakly feasible assignments cannot be too far from the optimal: if two advertisers are assigned to positions not in their order of bids, then either (i) the two advertisers have similar values for a click; or (ii) the click-through rates of the two slots are not very different, and hence in either case their relative order doesn't affect the social welfare very much.

Related work. Sponsored search has been a very active area of research in the last several years. For the basic model of Nash equilibria in such auctions see the papers by Edelman et al [3] and Varian [9], for a truthful auction see Aggarwal et all [1], and see the survey of Lahaie et al [7] for a general introduction. Since the original models, there has been much work in the area, exploring more complex models of click-through rates, taking into account budgets, analyzing dynamics, considering more complex models of incentives (such as vindictive bidding), etc. A lot of this work have been reported in the first four Workshops on Ad Auctions 2005 through 2008. Closest to our work is the paper Lahaie [6], that provides price of anarchy bounds on efficiency of equilibria, provided that the click-through-rate decays exponentially along the slots with a factor of δ.

Here we consider the simpler models of either click-through rates α_i that is a property of slot i independent of the advertiser, or separable click though rates, where the click through rate for bidder j in slot i can be expressed in a simple product form $\gamma_j \alpha_i$. For these models Edelman et al [3] and Varian [9] show that there exists Nash equilibria that are socially optimal. More precisely, they consider a restricted class of Nash equilibria called Envy-free equilibria or Symmetric Nash Equilibria, and show that such equilibria exists, and all such equilibria are socially optimal. In this class of equilibria, an advertiser wouldn't be better off after switching his bids with the advertiser just above him. Note that this is a stronger requirement than Nash, as an advertiser cannot unilaterally switch to a position with higher click-through by simply increasing their bid. Edelman et al [3] claim that if the bids eventually converge, they will converge to an envy-free equilibrium, otherwise some advertiser could increase his bid making the slot just above more expensive and therefore making the advertiser occupying it underbid him. They do not provide a formal game model that selects such equilibria. Vorobeychik and Reeves [11] use simulation to study stable equilibria.

Lahaie [6] also considers the problem of quantifying the social efficiency of an equilibrium. He proves a price of anarchy of $\min\{\frac{1}{\delta}, 1 - \frac{1}{\delta}\}$ provided that the click-through-rate decays exponentially along the slots with a factor of $\frac{1}{\delta}$. Feng et al [4] gives experimental evidence that click-through-rates decay exponentially. To prove the claimed bound, Lahaie develops a tool which is similar to ours. He proves π is a feasible allocation if and only if $\frac{v_{\pi(i)}}{v_{\pi(j)}} + \frac{\alpha_j}{\alpha_{i+1}} \geq 1$ for $1 \leq i \leq n-2$ and $j \geq i+2$.

In this paper, we consider a different restriction of Nash, we assume that bidders are conservative, in the sense that no bidder is bidding above their own valuation. We can justify this assumption by assuming that a new random bids can show up with a vanishingly small probability $\epsilon \to 0$. In equilibria that survive this perturbation, the bidders are conservative. Without any additional requirement Nash equilibria can have social welfare that is arbitrarily bad compared to the optimal social welfare. However, we show that Nash equilibria of conservative bidders is within a $\frac{1+\sqrt{5}}{2} \approx 1.618$ factor to the optimum. We assume only that the click-through-rates are separable (the product form) and are monotone.

2. PRELIMINARIES

We consider an auction with n advertisers and n slots (if there are less slots than advertisers, consider additional virtual slots with click-through-rate zero). Let v_i be the value that advertiser i has for one click and α_j be the click-through-rate of slot j. We will extend the results to separable click-through-rate at the end of the paper.

Assume that advertisers and slots ordered so that $v_1 \geq v_2 \geq \ldots \geq v_n$ and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. Given those parameters of the model, the mechanism of the Generalized Second Price Auction (GSP) is:

1. each advertiser submits a bid $b_i \geq 0$
2. the advertiser are sorted by their bids (ties are broken arbitrarily)
3. the highest slot is assigned to the advertiser with highest bid, the second highest slot to the one with second highest bid and so on.
4. the advertiser occupying slot i pays the bid of the advertiser occupying slot $i+1$. The advertiser occupying the last slot pays zero.

Let S_n be the set of permutations of n elements. We characterize the order of the advertisers in the slots using a permutation π so that $\pi(i)$ is the advertiser occupying slot i, which is the same of the advertiser with the i^{th} highest bid.

We define the **utility** of a user i when occupying slot j as given by $u_i = \alpha_j(v_i - b_{\pi(j+1)})$. Given a set of bids $b_1, \ldots, b_n$ we say that they constitute a **Nash equilibrium** if no advertiser can increase its own utility by changing his own bid. Suppose advertiser i is currently bidding b_i and occupying slot j. Changing his bid to something between $b_{\pi(j-1)}$ and $b_{\pi(j+1)}$ won't change the permutation π and therefore won't change the allocation nor his payment. So, he could try to increase his valuation by doing one of two things:

- increasing his bid to get a slot with a better click-through-rate. If he wants to get a slot $k < j$ he needs to overbid advertiser $\pi(k)$, say by bidding $b_{\pi(k)} + \epsilon$. This way he would get slot k for the price $b_{\pi(k)}$ per click, getting utility $\alpha_k(v_i - b_{\pi(k)})$.
- decreasing his bid to get a worse but cheaper slot. If he wants to get slot $k > j$ he needs to bid below advertiser $\pi(k)$. This way he would get slot k for the price $b_{\pi(k+1)}$ per click, getting utility $\alpha_k(v_i - b_{\pi(k+1)})$.

Therefore we say that b is a Nash equilibrium if the following equations hold:

$$\begin{aligned} & b_{\pi(1)} \geq b_{\pi(2)} \geq ... \geq b_{\pi(n)} \\ & \alpha_i(v_{\pi(i)} - b_{\pi(i+1)}) \geq \alpha_j(v_{\pi(i)} - b_{\pi(j)}) \qquad \forall j < i \qquad (1) \\ & \alpha_i(v_{\pi(i)} - b_{\pi(i+1)}) \geq \alpha_j(v_{\pi(i)} - b_{\pi(j+1)}) \qquad \forall j > i \end{aligned}$$

where π is the permutation defined by b. We say that π is a **feasible permutation** for α, v if there is a b that generates π and is a Nash equilibrium.

We measure the total quality of an equilibrium by the **social welfare**, which is defined as $\sum_j \alpha_j v_{\pi(j)}$. The optimal social welfare is naturally achieved when π is the identity permutation and [3] proves that there is always a Nash equilibrium that achieves that (in particular, allocation and payments in this equilibrium are equal to VCG). However not every Nash equilibrium is optimal, as we will see shortly. We are interested in quantifying the **price of anarchy** for this game, which is given by the maximum over all permutations that define Nash equilibria of $\sum_j \alpha_j v_j / \sum_j \alpha_j v_{\pi(j)}$.

2.1 Equilibria with Low Social Welfare

Even for two slots the gap between the best and the worse Nash equilibrium can be arbitrarily large. For example, consider two slots with click-through-rates $\alpha_1 = 1$ and $\alpha_2 = r$ and two advertisers with valuations $v_1 = 1$ and $v_2 = 0$. It is easy to check that the bids $b_1 = 0$ and $b_2 = 1 - r$ are a Nash equilibrium where advertiser 1 gets the second slot and advertiser 2 gets the first slot. The social welfare in this equilibrium is r while the optimal is 1. The price of anarchy is therefore $1/r$. Since r can be any number from 0 to 1, the gap between the optimal and the worse Nash can be arbitrarily large.

Notice however that this Nash equilibrium seems very artificial: advertiser 2 is exposed to the risk of negative utility: if advertiser 1 (or another advertiser) adds a bid somewhere between 0 and $1-r$ this imposes a negative utility on advertiser 2. Bidding $1 - r$ while having valuation 0 is accepting a lot of risk. We claim that if bidders are not willing to accept such risk (or accepts only a limited amount of such risk) then the price of anarchy is bounded.

3. CONSERVATIVE BIDDER EQUILIBRIA

We say an advertiser i is γ-conservative if $b_i \leq \frac{1}{\gamma} v_i$. So, generic advertisers are 0-conservative. We call conservative bidders the 1-conservative advertisers.

Note that if bidder i has to pay a price above v_i she has negative utility, and hence this cannot happen in a Nash equilibrium. A non-conservative bid $b_i > v_i$ can only be part of a Nash equilibrium if the resulting price p_i (the next smallest bid) is small enough $v_i \geq p_i$. In this case all bids b_i' in the range $(p_i, b_i]$ of user i result in the same outcome, and same payments, hence same utility. Now consider how the outcome and utility is effected if a new bid b^* is added to the system. If $v_i < b^* < b_i$ then user i remains to be assigned to the same slot, but will pay a rate b^* resulting in negative utility. In contrast, by bidding $b_i' = v_i$ bidder i does not effect its utility in the original game, and avoids the danger of negative utility when the bid b^* is added.

Given the parameters α, v, we say that b is a **conservative bidder equilibrium** if it is a Nash equilibrium and $b_i \leq v_i$ for all bidders i.

THEOREM 1. *A Nash equilibrium that remains an equilibrium in the game when a random bid is added with a small probability $\epsilon > 0$ is a conservative bidder equilibrium, and conservative bidder equilibria exists.*

PROOF. We argued above that Nash equilibria that survive a small enough perturbation are conservative bidder equilibria. To see that conservative bidder equilibria exist we use the equilibria of Edelman et al [3], where $b_1 = v_1$ and $b_i = \frac{1}{\alpha_{i-1}} \sum_{j \geq i-1} (\alpha_j - \alpha_{j+1}) v_{j+1}$ for $i > 1$ is clearly conservative. $\square$

For the remainder of the paper we consider conservative equilibria.

THEOREM 2. *For 2 slots, if all advertisers are γ-conservative, then the price of anarchy is bounded by $\frac{1+\gamma r(1-r)}{\gamma + r(1-\gamma)}$, where $r = \frac{\alpha_2}{\alpha_1}$*

In particular, taking $\gamma = 0$ we recover the $1/r$ bound for the general case and for $\gamma = 1$ we have a quadratic function with maximum equal to 1.25. It is not hard to see that this bound is limited for any $\gamma > 0$.

PROOF. We can suppose without loss of generality that $\alpha_1 = 1$, $\alpha_2 = r$ and $\alpha_1 v_1 + \alpha_2 v_2 = 1$, since what we are trying to prove is invariant under rescaling α or v. In any non-optimal Nash equilibrium $b_1 \leq b_2$ and by the Nash condition $r(v_1 - 0) \geq 1(v_1 - b_2)$ and by the conservative condition $b_2 \gamma \leq v_2$. Substituting $v_1 = 1 - r v_2$ in those two expressions and combining them to eliminate the b_2 term we get:

$$v_2 \geq \frac{1 - r}{\frac{1}{\gamma} - r(r - 1)} \qquad (2)$$

Therefore the social welfare in any non-optimal Nash is $\alpha_1 v_2 + \alpha_2 v_1 = 1 v_2 + r(1 - r v_2) \geq \frac{1+\gamma r(1-r)}{\gamma + r(1-\gamma)}$. $\square$

3.1 Weakly Feasible Assignments

Next we show that equilibria with conservative bidders satisfies the simple property mentioned in the introduction. We will call the assignments satisfying this property weakly feasible. In the next section we analyze the welfare properties of weakly feasible equilibria.

We start by showing that an assignment when no bidder i can increase its utility unless he bids above his valuation is in fact a Nash equilibrium in the usual sense (equations 1) in which $b_i \leq v_i$. For this equilibrium we still have the relations for $j > i$ as in equation 1 but for $j < i$, now we have:

$$\alpha_j(v_{\pi(i)} - b_{\pi(j)}) > \alpha_i(v_{\pi(i)} - b_{\pi(i+1)}) \Rightarrow b_{\pi(j)} > v_{\pi(i)}$$

that is equivalent to:

$$v_{\pi(i)} - b_{\pi(j)} \leq \frac{\alpha_i}{\alpha_j}(v_{\pi(i)} - b_{\pi(i+1)}) \text{ or } v_{\pi(i)} - b_{\pi(j)} < 0$$

and we can rewrite it as:

$$\begin{aligned} v_{\pi(i)} - b_{\pi(j)} &\leq \max\left\{\frac{\alpha_i}{\alpha_j}(v_{\pi(i)} - b_{\pi(i+1)}), 0\right\} \\ &= \frac{\alpha_i}{\alpha_j}(v_{\pi(i)} - b_{\pi(i+1)}) \end{aligned}$$

since $v_{\pi(i)} \geq b_{\pi(i)} \geq b_{\pi(i+1)}$. So it is a Nash equilibrium in the standard sense with the additional constraints that $b_i \leq v_i$.

The equations 1 are not very easy to work with, since they are not very symmetric and they depend on b. We propose a cleaner form of representing an equilibrium that just uses α, v and the permutation π. Although it is a weaker property it still captures most of the trade-offs:

1. if values v_i are very close then the order of the bidders doesn't influence the social welfare that much
2. if values v_i are very well separated, then permutations that would produce a bad social welfare are not feasible because they violate Nash constraints

Theorem 3. *Given v, α and a Nash permutation π, if $i < j$ and $\pi(i) > \pi(j)$ then:*

$$\frac{\alpha_j}{\alpha_i} + \frac{v_{\pi(i)}}{v_{\pi(j)}} \geq 1 \tag{3}$$

in particular, $\frac{\alpha_j}{\alpha_i} \geq \frac{1}{2}$ or $\frac{v_{\pi(i)}}{v_{\pi(j)}} \geq \frac{1}{2}$.

Proof. Since it is a Nash equilibrium bidder in slot j is happy with his condition and don't want to increase his bid to take slot i, so:

$$\alpha_j(v_{\pi(j)} - b_{\pi(j+1)}) \geq \alpha_i(v_{\pi(j)} - b_{\pi(i)})$$

since $b_{\pi(j+1)} \geq 0$ and $b_{\pi(i)} \leq v_{\pi(i)}$ then:

$$\alpha_j v_{\pi(j)} \geq \alpha_i(v_{\pi(j)} - v_{\pi(i)})$$

□

Inspired by the last theorem, given parameters α, v we say that permutation π is *weakly feasible* if equation 3 holds for each $i < j$, $\pi(i) > \pi(j)$. From Theorem 2 we know that:

Corollary 4. *Given α, v, any permutation corresponding to a Nash equilibrium with conservative bids is weakly feasible.*

Our main results follow from analyzing the price of anarchy ratio $\sum_j \alpha_j v_j / \sum_j \alpha_j v_{\pi(j)}$ over all weakly feasible permutations π. Before proceeding to the main result. we re-prove the bound in [6] for the conservative case.

Theorem 5. *If $\frac{\alpha_i}{\alpha_{i+1}} \geq \delta > 1$ for all i, then if π is a weakly feasible permutation, then the price of anarchy is bounded by $1 - \frac{1}{\delta}$, i.e.:*

$$\sum_i \alpha_i v_{\pi(i)} \geq (1 - \delta^{-1}) \sum_i \alpha_i v_i$$

Proof. If $\pi(i) > i$ then there is some $j > i$ such that $\pi(j) \leq i$ (by the pigeonhole principle, since there are only $i-1$ slots with index $< i$, so at least one of the first i bidders must occupy one slot after i). So, as $\pi(j) \leq i < \pi(i)$ and $j > i$ we can apply our relation:

$$v_{\pi(i)} \geq \left(1 - \frac{\alpha_j}{\alpha_i}\right) v_{\pi(j)} \geq \left(1 - \frac{\alpha_j}{\alpha_i}\right) v_i \geq (1 - \delta^{-1}) v_i$$

where the first inequality is that of Theorem 3. The theorem follows almost directly:

$$\begin{aligned} \sum_i \alpha_i v_{\pi(i)} &= \sum_{\pi(i) \leq i} \alpha_i v_{\pi(i)} + \sum_{\pi(i) > i} \alpha_i v_{\pi(i)} \\ &\geq \sum_{\pi(i) \leq i} \alpha_i v_i + \sum_{\pi(i) > i} \alpha_i v_i (1 - \delta^{-1}) \\ &\geq (1 - \delta^{-1}) \sum_i \alpha_i v_i \end{aligned}$$

□

3.2 The Main Results

Here we present the bound on the price of anarchy for weakly feasible permutations, and hence for GSP for conservative bidders. Our main result is that it is bounded by 1.618. As a warm-up we will prove that it is bounded by 2, since the proof is easier and captures the main ideas. We will prove this bound for weakly feasible permutations and it will automatically be deduced to a bound for feasible permutations. Notice that the weakly feasible permutation nicely capture the fact that if advertisers i and j are in the "wrong relative position" (i.e. different to the one in the optimal) then either their values are close (within a factor of 2) or their click-through-rates are close (within a factor of 2).

Theorem 6. *For conservative bidders, the price of anarchy for GSP is bounded by 2.*

Proof. We will prove it by induction on n that all weakly feasible permutations result in social welfare at most of factor of 2 less than the maximum possible. For 2 advertisers and 2 slots we know that the worst possible social welfare for a weakly feasible permutation is at most a 1.25 fraction of

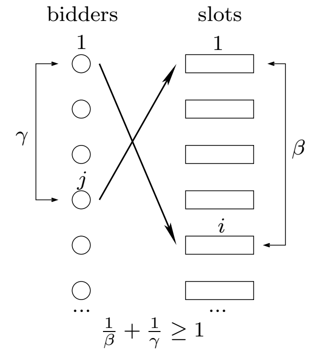

Figure 1: Allocation of slots in the proof of Theorems 5 and 6

the optimum. So, now we need to prove the inductive step. Consider parameters v, α and a weakly feasible permutation π. Let $i = \pi^{-1}(1)$ be the slot occupied by the advertiser of higher value and $j = \pi(1)$ be the advertiser occupying the first slot (as shown in Figure 1). If $i = j = 1$ then we can apply the inductive hypothesis right away. If not, equation 3 tells us that: $\frac{\alpha_i}{\alpha_1} \geq \frac{1}{2}$ or $\frac{v_j}{v_1} \geq \frac{1}{2}$. Suppose $\frac{\alpha_i}{\alpha_1} \geq \frac{1}{2}$ and consider an input with slot i and advertiser 1 deleted. This input has $n-1$ advertisers and $n-1$ slots and the permutation π restricted to those is still weakly feasible, so by the inductive hypothesis:

$$\sum_{k \neq i} \alpha_k v_{\pi(k)} \geq \frac{1}{2}(\alpha_1 v_2 + ... + \alpha_{i-1} v_i + \alpha_{i+1} v_{i+1} + ... + \alpha_n v_n)$$
$$\geq \frac{1}{2}(\alpha_2 v_2 + ... + \alpha_i v_i + \alpha_{i+1} v_{i+1} + ... + \alpha_n v_n)$$

therefore:

$$\sum_k \alpha_k v_{\pi(k)} = \alpha_i v_1 + \sum_{k \neq i} \alpha_k v_{\pi(k)} \geq \frac{1}{2}\alpha_1 v_1 + \frac{1}{2}\sum_{k>1} \alpha_k v_k$$

If $\frac{v_j}{v_1} \geq \frac{1}{2}$ we just do the same but deleting slot 1 and advertiser j from the input. $\square$

Now, we prove the tighter result.

THEOREM 7. *For conservative bidders, the price of anarchy is bounded by* $\frac{1+\sqrt{5}}{2} \approx 1.618$.

PROOF. As before, we prove the conclusion for all weakly feasible permutations. We use here a dynamical systems argument: we define a sequence of values r_k so that we can prove that for k slots social welfare is at least an r_k fraction of the optimum, and prove that r_k converges to the desired bound. Let $r_2 = 1.25$ and suppose we have $r_2, r_3, ..., r_{n-1}$ and that this property holds for them. Let's calculate some "small" value of r_n so that the property still holds.

Again, consider parameter α, v, a weakly feasible permutation π and let's assume $i = \pi^{-1}(1)$ and $j = \pi(1)$ (as shown in Figure 1). If $i = j = 1$, this is an easy case and it is straightforward to see that in this case the price of anarchy can be bounded by r_{n-1}. If not, assume without loss of generality that $i \leq j$ (since equation 3 is symmetric in α and v we can just interchange the roles of them in the proof if $i > j$). Let $\beta = \frac{\alpha_1}{\alpha_i}$ and $\gamma = \frac{v_1}{v_j}$. We know that $\frac{1}{\beta} + \frac{1}{\gamma} \geq 1$. Following the lines of the proof of the last theorem we have:

$$\sum_k \alpha_k v_{\pi(k)} = \alpha_i v_1 + \sum_{k \neq i} \alpha_k v_{\pi(k)} \geq$$
$$\geq \frac{1}{\beta}\alpha_1 v_1 + \frac{1}{r_{n-1}}\left(\sum_{k=2}^{i} \alpha_{k-1} v_k + \sum_{k=i+1}^{n} \alpha_k v_k\right) \geq$$
$$= \frac{1}{\beta}\alpha_1 v_1 + \frac{1}{r_{n-1}}\left[\sum_{k=2}^{i} (\alpha_{k-1} - \alpha_k) v_k + \sum_{k>1} \alpha_k v_k\right] \geq$$
$$\geq \frac{1}{\beta}\alpha_1 v_1 + \frac{1}{r_{n-1}}(\alpha_1 - \alpha_i) v_i + \frac{1}{r_{n-1}}\sum_{k>1} \alpha_k v_k \geq$$

Now, we can use $i \leq j$ to say: $v_i \geq v_j = \frac{1}{\gamma} v_1 \geq \left(1 - \frac{1}{\beta}\right) v_1$.

$$\sum_k \alpha_k v_{\pi(k)} \geq \left[\frac{1}{\beta} + \frac{1}{r_{n-1}}\left(1 - \frac{1}{\beta}\right)^2\right] \alpha_1 v_1 + \frac{1}{r_{n-1}}\sum_{k>1} \alpha_k v_k$$

So, we would like to find some r_n such that we can say that $\sum_k \alpha_k v_{\pi(k)} \geq \frac{1}{r_n}\sum_k \alpha_k v_k$ for all $\beta \geq 1$, so we would like to have:

$$\frac{1}{r_n} \leq \min\left\{\frac{1}{r_{n-1}}, \frac{1}{\beta} + \frac{1}{r_{n-1}}\left(1 - \frac{1}{\beta}\right)^2\right\}$$

for any $\beta \geq 1$. But notice some other bound we can get is:

$$\sum_k \alpha_k v_{\pi(k)} \geq \frac{1}{\gamma}\alpha_1 v_1 + \frac{1}{r_{n-1}}\sum_{k>1} \alpha_k v_k$$
$$\geq \left(1 - \frac{1}{\beta}\right)\alpha_1 v_1 + \frac{1}{r_{n-1}}\sum_{k>1} \alpha_k v_k$$

by following the lines of the proof of last theorem, but removing slot 1 and advertiser j in the inductive step. So another alternative is to get:

$$\frac{1}{r_n} \leq \min\left\{\frac{1}{r_{n-1}}, 1 - \frac{1}{\beta}\right\}$$

for every $\beta \geq 1$. So if we can get $1/r_n$ bounded by the maximum of those two quantities, we are done. Summarizing that, we need:

$$r_n \geq \max\left\{r_{n-1}, \left[\max\left\{1 - \frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{r_{n-1}}\left(1 - \frac{1}{\beta}\right)^2\right\}\right]^{-1}\right\}$$

for all $\beta \geq 1$.

Now we need to evaluate for which value of $\frac{1}{\beta} \in (0, 1]$ the expression $\max\left\{1 - \frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{r_{n-1}}\left(1 - \frac{1}{\beta}\right)^2\right\}$ has its minimum. The minimum can be in two points: the minimum of the quadratic function or the intersection between those two functions. They intersect at $\frac{1}{\beta} = -r + 1 + \sqrt{r^2 - r}$ (where r stands for r_{n-1}) and the quadratic minimum is at $1 - \frac{1}{2}r$. So, for $r \geq \frac{4}{3}$, the minimum occurs in the intersection and

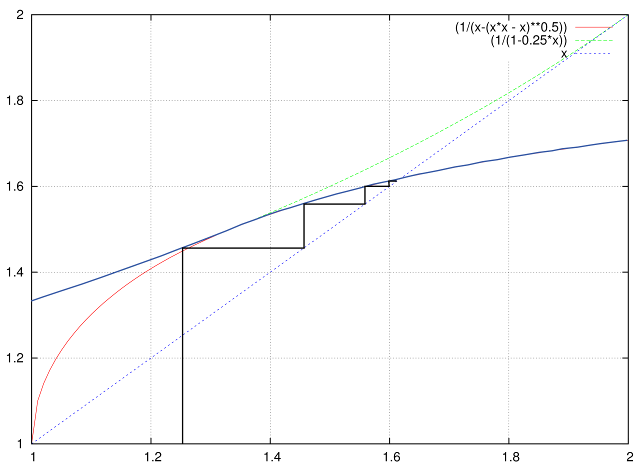

Figure 2: Sequence of values r_k that are an upper bound of the price of anarchy for k slots

for $r < \frac{4}{3}$, it occurs in the quadratic minimum. So:

$$r_n = \begin{cases} \left(1 - \frac{r_{n-1}}{4}\right)^{-1} & , r_{n-1} < \frac{4}{3} \\ \left(r_{n-1} - \sqrt{r_{n-1}^2 - r_{n-1}}\right)^{-1} & , r_{n-1} \geq \frac{4}{3} \end{cases}$$

since we want the smallest possible ratio. This allows to define r_k recursively from $r_2 = 1.25$ and it is easy to see that the sequence monotonically converges to the fixed point of that function which is the golden ration $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, as shown in Figure 2. This happens because the function that maps r_{n-1} to r_n is non-decreasing and has a fixed point in φ, so if $r_{n-1} \leq \varphi$ then $r_n \leq \varphi$.

□

To illustrate how symmetric and easy to work with this new formulation is, we also add the following result:

THEOREM 8. *The worse possible price of anarchy among all possible parameters n, α, v and all possible weakly feasible permutations π occurs when π is a simple cycle, i.e, exist $\{x_1, ..., x_n\} = \{1, ..., n\}$ such that $\pi(x_i) = x_{i+1}$ for $i < n$ and $\pi(x_n) = x_1$.*

PROOF. If π is weakly feasible but is not a simple cycle, then we can decompose this permutation as a product of two disjoint permutations $\pi = \pi_1\pi_2$ with supports N_1 and N_2, i.e, π_i moves the bidders and slots with indices in N_i. So, we have:

$$\frac{\sum_k \alpha_k v_{\pi(k)}}{\sum_k \alpha_k v_k} = \frac{\sum_{k\in N_1} \alpha_k v_{\pi(k)} + \sum_{k\in N_2} \alpha_k v_{\pi(k)}}{\sum_{k\in N_1} \alpha_k v_k + \sum_{k\in N_2} \alpha_k v_k} \leq$$
$$\leq \max\left\{\frac{\sum_{k\in N_1} \alpha_k v_{\pi(k)}}{\sum_{k\in N_1} \alpha_k v_k}, \frac{\sum_k \alpha_{k\in N_2} v_{\pi(k)}}{\sum_{k\in N_2} \alpha_k v_k}\right\}$$

and π_i is weakly feasible over N_i (i.e., the restricted input of slots with indices in N_i and advertisers with indices in N_i). □

3.3 Extension to separable click-through-rates

So far, we have considered that the click-through-rates of advertiser i placed on slot j depends only on the slot in which he is placed. A more general model called **separable click-through-rates** assumes it depends on a product of two factors: one depending on the bidder and other depending on the slot. Let's say that if advertiser i is placed on slot j, it will get click-through-rate $\gamma_i\alpha_j$ where γ_i is some "quality factor" attributed to each advertiser. The generalization of Second Price Auction for this setting ranks the advertisers in order of $\gamma_i b_i$ and charges an advertiser the minimum value it needed to bid to conserve his position. For example, if π is the permutation defined by sorting $\gamma_i b_i$ (i.e, $\pi(k)$ is the advertiser with the k^{th} highest value of $\gamma_i b_i$) then we charge advertiser $\pi(j)$ the amount of: $b_{\pi(j+1)}\gamma_{\pi(j+1)}/\gamma_{\pi(j)}$.

In this setting the utility of bidder i assigned to slot j is $u_i = \gamma_i\alpha_j\left(v_i - \frac{b_{\pi(j+1)}\gamma_{\pi(j+1)}}{\gamma_i}\right)$ and the social welfare is given by $\sum_k \alpha_k\gamma_{\pi(k)}v_{\pi(k)}$. Consider that $\alpha_1 \geq ... \geq \alpha_n$ and that $\gamma_1 v_1 \geq ... \geq \gamma_n v_n$. The definition of Nash equilibrium is analogous. Notice we can obtain a result very similar with Theorem 3 just by repeating the same calculations for this model:

THEOREM 9. *Given v, α, γ and a feasible permutation π (a permutation from a Nash equilibrium) in the separable click-through-rate model, if $i < j$ and $\pi(i) > \pi(j)$ then:*

$$\frac{\alpha_j}{\alpha_i} + \frac{\gamma_{\pi(i)}v_{\pi(i)}}{\gamma_{\pi(j)}v_{\pi(j)}} \geq 1 \quad (4)$$

PROOF. Since advertiser $\pi(j)$ can't increase his utility by taking slot i, we have that:

$$\gamma_{\pi(j)}\alpha_j\left(v_{\pi(j)} - \frac{b_{\pi(j+1)}\gamma_{\pi(j+1)}}{\gamma_{\pi(j)}}\right) \geq \gamma_{\pi(j)}\alpha_i\left(v_{\pi(j)} - \frac{b_{\pi(i)}\gamma_{\pi(i)}}{\gamma_{\pi(j)}}\right)$$

using that $b_{\pi(j+1)} \geq 0$ and $b_{\pi(i)} \leq v_{\pi(i)}$ we get the desired result. □

And all other results follow with almost no change.

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