Gross substitutes: Combinatorial structure and algorithms
Tutorial proposal for EC'18

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1 Theme

Many fundamental results in market design with indivisible goods (combinatorial auctions, two-sided matchings, exchanges) crucially rely on *substitutability* of the players' preferences. These include computational tractability of welfare maximization; nice incentive and revenue properties of the VCG mechanism; existence of market clearing prices that support the welfare-maximizing allocation in equilibrium; and existence of stable matchings. The diversity of settings, applications and disciplines in which substitutability comes up is highly intriguing, and it turns out to have deep mathematical roots. A concept equivalent to subsitutability ($M^\natural$-concavity) was independently discovered in the field of discrete convex analysis as a natural generalization of convex functions to the integer lattice ([Murota, 2003] and [Murota and Shioura, 1999]).

Substitutability is captured by the class of *gross substitutes valuations*. This class has received less attention from the Computer Science community compared to higher classes in the valuation hierarchy, such as submodular and subadditive valuations. More attention is merited however: Viewing gross substitutes through the algorithmic lens reveals many open questions, related to characterization, representation and learning, computationally efficient algorithms for welfare maximization, market coordination and pricing, robustness to limited complementarities, and relations to other valuation classes. The importance of these questions stems from the centrality of the substitutability assumption. Recently there has been a flurry of interest in these and related problems (see, e.g., [Baldwin and Klemperer, 2017; Ben-Zwi, 2017; Cohen-Addad et al., 2016; Deng et al., 2017; Milgrom, 2017; Ostrovsky and Paes Leme, 2015; Paes Leme and Wong, 2017; Roughgarden and Talgam-Cohen, 2015; Roughgarden et al., 2017; Soma and Yoshida, 2017]). A recent survey summarizes the current state-of-the-art [Paes Leme, 2017].

The purpose of this tutorial is to give an overview of substitutability with an eye towards open directions, and to provide a starting point for research in these directions.

2 Tutorial Organization

In each part of the tutorial, we survey "classic" literature, recent developments and open problems.

1. **Combinatorial structure:** After briefly presenting motivating applications, we survey the main definitions of gross substitutability and sketch their equivalence. We focus on algorithmic aspects of the various definitions. The concept of gross substitutes valuations was introduced by Kelso and Crawford [1982] as a sufficient condition for the existence of a Walrasian equilibrium in economies with indivisible goods. The proof is algorithmic in nature: gross substitutes is exactly the condition that enables a natural price adjustment procedure

– known as Walrasian tâtonnement – to converge to equilibrium. In addition, gross substitutes are precisely the class of valuations for which *demand queries* can be computed via an ascending greedy algorithm. The same concept was also introduced independently in other communities under different names, including as $M^\natural$-concave functions [Murota, 2003] and [Murota and Shioura, 1999]. This class of functions is important since it embodies discrete concavity – local maximizers correspond to global maximizers, opening the way for discrete convex analysis. Gross substitutes are also equivalent to matroidal maps (the real-valued equivalent of matroids) and valuated matroids [Dress and Wenzel, 1992; Dress and Terhalle, 1995].

2. **Algorithms:** We discuss algorithms for the welfare problem (computing an optimal allocation of a set of items when agents have gross substitutes valuations) as well as the related problem of computing Walrasian prices. We discuss approximation schemes based on the tâtonnement procedure, linear programming approaches, and purely combinatorial, strongly polynomial time algorithms. In particular we present the useful matching construction by Murota Murota [1996a,b], connect Walrasian prices to weighted shortest paths, and discuss the connection to the general matroid intersection problem [Paes Leme and Wong, 2017; Murota and Tamura, 2003].

3. **Extensions:** How realistic is the substitutability assumption? Real-world experience often reveals near-optimal performance of greedy algorithms, in particular in auction settings [Milgrom, 2017, Section 4.3] (see also [Echenique et al., 2011]). This may hint that valuations in practice tend to be "approximately" gross substitutes. We summarize recent attempts at formulating ϵ-approximate gross substitutes. A successful formulation may open the way for extending the above algorithms and mechanisms beyond gross substitutes, to valuations with limited complements. We also discuss a model generalizing gross substitutes valuations and substitutable matching preferences [Hatfield and Milgrom, 2005], and an extension based on *demand types* [Baldwin and Klemperer, 2017].

   Proposed Length: 3 hours.

Further topics (time-allowing): Operations that preserve substitutability (convolution, restriction, endowment). Discrete derivatives, discrete Hessian, ultrametrics and substitutability trees. Discrete Fenchel duality. Substitutable divisible goods.

2.1 Organizers

Renato Paes Leme is a research scientist at Google Research New York. He is broadly interested in algorithm design, especially for problems on the interface between Economics and Computation. He graduated in 2012 with a PhD in Computer Science from Cornell University where he was advised by Éva Tardos. He spent one year as a postdoc at Microsoft Research Silicon Valley working with Moshe Babaioff and Noam Nisan. Website: `www.renatoppl.com`. Email: `renatoppl@google.com`.

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