# Reservation Exchange Markets for Internet Advertising [*]

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## ABSTRACT

Internet display advertising industry follows two main business models. One model is based on *direct deals* between publishers and advertisers where they sign legal contracts containing terms of fulfillment for a future inventory. The second model is a spot market based on auctioning page views in real-time on advertising exchange (AdX) platforms such as DoubleClick's Ad Exchange, RightMedia, or AppNexus. These exchanges play the role of intermediaries who sell items (e.g. page-views) on behalf of a seller (e.g. a publisher) to buyers (e.g., advertisers) on the opposite side of the market. The computational and economics issues arising in this second model have been extensively investigated in recent times.

In this work, we consider a third emerging model called *reservation exchange market*. A reservation exchange is a two-sided market between buyer orders for blocks of advertiser' impressions and seller orders for blocks of publisher' page views. The goal is to match seller orders to buyer orders while providing the right incentives to both sides. In this work we first describe the important features of mechanisms for efficient reservation exchange markets. We then address the algorithmic problems of designing revenue sharing schemes to provide a fair division between sellers of the revenue collected from buyers.

A major conceptual contribution of this work is in showing that even though both clinching ascending auctions and VCG mechanisms achieve the same outcome from a buyer perspective, however, from the perspective of revenue sharing among sellers, clinching ascending auctions are much more informative than VCG auctions.

## 1. INTRODUCTION

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The universe of internet advertisement is divided in two big worlds: *search ads* and *display ads*. At first glance, they look very similar: both sell impressions using variants of the second price auction with reserves. A closer look, however, reveals that they are very different: in *search ads*, the platform (Google or Bing, for example) is both the *auctioneer* and the *seller*, i.e., it sells inventory in their own properties. This makes it a one-sided market design problem, or in other words, the designer needs to reason only about the incentives of the buyers. In *display ads*, the platform is auctioning inventory not owned by them, turning it into a two-sided market design problem, where incentives for buyers (advertisers) and seller (publishers, such as websites, blogs and news portals) need to be balanced.

Designing practical markets for display ads is challenging, since the theory of market design is much more developed for one-sided markets (there are tools like VCG, Myerson's optimal auction, ...) while for two-sided markets, what classic auction theory offers are mainly impossibility results, such as the Myerson-Satterthwaite Impossibility Theorem [**?**].

One other complicating factor in the display ads is that while we are used to think of internet advertisement in the form of auctions, auctions are only the tip of the iceberg. The most premium inventory is sold via the *reservations market* (also called *direct deals* or *guaranteed contracts*. In this method, a publisher and an advertiser make a deal to allocate a certain number of impressions over a certain period, for a pre-specified price per impression. This deal is made offline in advance for a future inventory. Direct deals are known to suffer from inefficiencies for two reasons: first is the manual nature of formation of these contracts which allows a publisher to sign deals with a small number of advertisers, thus creating allocation inefficiencies. The second reason is the manual negotiation between buyers and publishers which incurs a huge cost and lowers the overall efficiency.

Auctions, on the other hand, are fully automated and don't suffer from any of those inefficiencies. On the other hand, they can't guarantee to buyers and sellers the certainty that reservations can. For example, a brand launching a large campaign to advertise a new product certainly benefits from the certainty (both in terms of cost and volume of impressions) provided by reservation contracts.

The idea of *automated reservation market* is to overcome the shortcomings of both auctions (real-time spot market) and traditional reservations (offline negotiation). Such market would allow sellers and buyers to transact for a bulk inventory in advance. This is inspired by a number of recent two-sided markets for online advertising, e.g., an exchange for future contracts[1]. In particular, we study a two-sided market, which we call a *reservation market*, where publishers can post offers for blocks of ad slots characterized by parameters like supply level, reserve price, and their preference for a set of advertisers. Advertisers post requests for ad slots defined by parameters like valuation, demand, and targeting constraints specifying where and when they want to show their ads. The role of the reservation mechanism is to match the seller and buyer orders that attains some economic objectives.

Note that, unlike ad exchange markets that offers impressions available on the spot, the reservation market offers guaranteed deals for an inventory available in the future. Moreover, the reservation market brings together multiple publishers and advertisers with the goal of reducing the intermediation costs and the underlying inefficiencies of one-to-one deals, also by selling bundles of inventories from different publishers.

In this work, we start the investigation of the economics and algorithmic principles that are central to these reservation markets. The major questions we address in this work are: What features and incentive properties form the basis of a successful reservation market? What are the economic objectives of a reservation market? What are the revenue sharing policies that we can employ? What are the algorithmic problems we need to address in the design of reservation mechanisms?

## 1.1 Our contribution

Our main contribution is to propose a formal model of reservation exchange market and discuss what are desirable properties (referred to as *axioms*) for this market. Secondly, we propose two specific mechanisms that help us understand the extent to which some of the aforementioned axioms can be simultaneously achieved. We also provide several algorithmic results for the two mechanisms that we study.

### *Axioms for a reservation exchange market.*

In Section **??**, we provide a simple and clean abstraction of reservation exchange markets for display ads as a two-sided matching market with buyer orders on one side and seller orders on the other side and in Section **??** we identify a list of axioms that we wish any mechanism for these markets to approximately satisfy.

The first axiom for the reservation mechanism that we discuss is the *efficiency of the market,* i.e., the social welfare of all agents of the market. The agents of the market are sellers and buyers, both with quasi-linear utilities. Buyers aim at maximizing their utility, i.e., the total value of the inventory received minus price. Sellers aim at maximizing revenue minus reservation price. The mechanism decides on the allocation to buyer orders of the inventory supplied from each seller, a payment to be charged to each buyer and a distribution of the revenue among sellers.

*Individual rationality* (IR) requires that participating in the mechanism is beneficial to all agents. *Incentive compatibility* (IC) requires that truthfully reporting one's preferences to the mechanism is the best strategy for each agent, independently from what the other agents report.

*Budget Balance* (BB) states the payments of the advertisers must entirely and exclusively be transferred to the publishers, i.e., the buyers and the sellers are allowed to trade without leaving to the mechanism any share of the payments, and without the mechanism adding money into the market. This axiom might appear strange at first glance, but it reflects a business practice common to most exchanges, which is of the exchange to get a fixed cut (typically called *revenue sharing margin*) of the seller's revenue. The reasoning behind this rule is that sellers have the option to send their inventory to different exchanges and keeping the revenue sharing margin fixed helps the exchange to be perceived as fair and hence attract more seller's inventory. Since fixed margins are an industrial standard in the ads world, any practical mechanism must implement some of that. Fixed margins are equivalent to budget-balance applied to bids rescaled by the revenue sharing margin up to rescaling bids.

An ideal goal in a reservation exchange is to design IR, IC, BB mechanisms that maximize the social welfare of all agents in the market. Unfortunately, Myerson and Satterthwaite [**?**] proved impossibility for an IR, IC, and BB mechanism that maximizes social welfare in such a market. The direction we pursue in this work is to ensure full efficiency and incentive compatibility for buyers as the most desirable goal for advertiser. This can be achieved for one-side markets by the family of Vickrey-Clarke-Groves (VCG) mechanisms [**?**, **?**]. As extensively discussed in this work, the main problem with the vanilla VCG allocation is that it does not offer any good incentives to sellers, e.g., VCG can match fungible inventories from different sellers to buyers that offer very different payments thus producing the feeling that revenue is unfairly distributed among sellers.

An alternative to enforcing incentive compatibility for sellers is to design a mechanism that leads to a fair distribution of mechanism' revenue among sellers. Envy-free allocations [**?**] and other notions of market equilibria have often been considered in markets that cannot achieve full efficiency with truthful allocations. This leads to our third axiom as follows: sellers should not envy each other with respect to the revenue that is received from the mechanism.

Most of the first part of the paper will be devoted to the discussion of which definition of *envy-freeness among sellers* is meaningful for reservation exchange markets. Our conclusion is that such a definition should crucially rely on the buyer-seller transactions that can arise in an efficient allocation, while it should disregard revenue that can only be obtained from allocations with suboptimal social welfare. With this aim, when introducing our notion of envy-freeness between sellers, we define the concept of *clinching graph* as the collection of buyer-seller transactions that can arise in a VCG allocation. We conveniently define and compute the clinching graph by resorting to the implementation of VCG through an ascending clinching auction.

A major insight of this paper is that while the usual description of VCG payments as externalities imposed by agents on others offers little clue on how to split the proceeds of the auction among the sellers, the alternative description of VCG as an ascending auction (Ausubel's clinching framework [**?**]) provides additional structure obtained from the execution of the auction that can be exploited to design revenue sharing schemes. The clinching auction returns not only bundle prices, but the order in which each item was sold and the price at which the sale occurred. In an ideal case,

[1] http://www.massexchange.com/

whenever a *clinch* happens, if the clinching auction points to a unique item to be clinched, then there is a unique way to split the revenue among the sellers (and in this case clinching auctions capture the full information in how to split the revenue). However, sometimes, when a clinch happens, there are multiple items that can be used for that clinch. This is precisely the case when the clinching auction, even though it provides more information than VCG, it doesn't lead to a unique revenue sharing scheme, and we rely on a notion of envy-freeness for the revenue sharing scheme.

Finally, we discuss further desirable properties of reservation markets as additional axioms, and study our proposed mechanisms for their satisfaction of these axioms. These axioms are stability properties that prevent the market to be manipulated from buyers or sellers. We define the concept of *buyer monotonicity (BM)* as the property that the revenue of all sellers does not decrease when new buyer orders are presented. A second property called *seller monotonicity (SM)* states that the increase of the reservation price of a seller is not responsible of the decrease of the revenue of another seller.

### *Algorithmic results.*

We restrict our attention to buyer incentive-compatible efficient mechanisms based on VCG allocations. For all these mechanisms truth-telling is a dominant strategy for buyers. The major issue we face is to complement the VCG mechanism with a suitable envy-free revenue sharing scheme between sellers. Our first result is actually a negative result:

- There exists no efficient revenue sharing scheme that is both envy-free and budget balance. We actually demonstrate that any envy-free revenue sharing scheme cannot distribute more than a $\sqrt{3}-1+\delta$ share of the total revenue, for any $\delta > 0$.

Given the impossibility result above, we investigate the possibility of finding good trade-offs between budget balance or envy freeness. Relaxing one of these two constraints imply that either the mechanism is able to distribute a guaranteed share of the total revenue or that any seller has only limited envy of any other seller. With this goal in mind, we propose two revenue sharing schemes: i) a *revenue sharing by the clinching auction (CA)*, and ii) a *revenue sharing by the eating mechanism (EM)*.

For the CA revenue sharing scheme, we prove the following desirable properties:

- CA is budget balance.
- CA is 1/2-envy free and this bound is tight.
- CA is budget monotone and seller monotone.

Finally, for the EM revenue sharing scheme, we prove that following three results:

- EM is envy-free.
- EM is at most $\frac{11+\epsilon}{12+\epsilon}$ budget balance, for any $\epsilon > 0$.
- EM is at least $\frac{e-1}{e}$-budget balance.

## 1.2 Related work

### *Double Auctions.*

Double auctions are special cases of two-sided markets with unit-supply buyers and sellers. Myerson and Satterthwaite [?] proved that it is impossible to obtain an IR, Bayesian IC[2], and weak BB [3] mechanism to maximize social welfare in double auctions. Since then, much of the literature has focused on trading off social welfare for buyers and sellers, incentive compatibility and budget balance for double auctions [?, ?, ?, ?, ?]. The seminal work on double auctions [?] shows that efficiency for both buyers and sellers can actually be achieved asymptotically in large markets. In the context of one-shot auctions, optimal auctions for two-sided settings has been studied by [?], following the Nobel-prize winning work of [?]. The problem of finding the right trade-offs between social welfare, IC and BB is largely open for two-sided markets that model reservation exchanges. In this work we investigate two-sided markets that achieve IC or buyers and envy-freeness for sellers. This follows a line of work that looks at envy-freeness and other market equilibria if social welfare cannot be optimized truthfully [?, ?, ?]. Recently, this literature has also been adopted to design the optimal revenue sharing double auctions in the context of advertising exchanges [?]. In this paper, we focus on two-sided markets where multiple buyers are allocated to multiple sellers and the allocation and pricing are done differently. Other than online advertising systems, optimal two-sided markets can be applied to online and offline retailers and e-commerce websites like Amazon and Ebay. A very recent paper by [?] studies EBay's double auction problem, but their setting is different from this paper as they consider one buyer and multiple sellers, and explore approximately optimal pricing schemes for this setting.

### *Clinching Ascending Auctions.*

One fundamental component of this work is the use of the structure that can be obtained from the execution of Ausubel's clinching auction [?] in designing revenue sharing schemes for the sellers. The clinching auction has been very successful in a variety of scenarios: designing auctions with budget constraints [?, ?, ?, ?], designing online auctions [?], extracting revenue in settings with budgets [?, ?]. The current paper adds to this line of work by showcasing another application of the clinching framework.

### *Cooperative games.*

Cooperative game theory may provide insights for modeling the fair sharing of revenue in the ad reservation exchange market. Shapley value [?] is a widely adopted notion of fair division between agents of the value of a game. It is however hard to extend this concept to our model since a crucial axiom of Shapley value (summability) does not hold in our case. Similar difficulties can also be found while trying to design a revenue sharing scheme that results in an attribution that lies in the core of a game [?]. On the positive side, we mention that the revenue sharing scheme by the clinching auction we present resembles Shapley values since it is defined as the expected revenue obtained over all possible seller permutations.

[2]Bayesian incentive compatibility is a less restrictive form of incentive compatibility

[3]Weak budget balance allows the mechanism to retain a share of the payments while not subsidizing the market.

*Market Equilibria.*

Several notions of equilibrium in markets have been studied. In a Walrasian equilibrium, we have item prices such that every agent receives a bundle of items that maximizes her utility, the market clears, and the corresponding outcome is efficient. However, except from very special cases (e.g. unit demand buyers), it can't be converted to a mechanism that is incentive-compatible for the buyer [**?**, **?**]. Envy-freeness for buyers is also a concept widely used to characterize the stability of allocations. We do not survey here the extensive literature on this topic. However, we notice that we instead adopt the notion of envy-freeness to characterize fair revenue sharing schemes between sellers.

Reservation-based Internet advertising has also previously considered with more optimization-related questions than mechanism design questions. Examples of this line of work that is quite unrelated to the scope of this paper can be found in [**?**, **?**, **?**]. Markets that combine characteristics of the spot market and of direct deals between publishers and sellers have been also considered in [**?**] with the goal of maximizing the revenue of one single publisher.

# 2. PRELIMINARIES

We consider a two-sided market, referred to as a *reservation exchange market*, consisting of a set $B$ of buyers and a set $S$ of sellers. Each seller $s_i \in S$ holds a supply of $\ell_i$ units of an *indivisible* good and has a reserve price $\rho_i$ for each unit of those goods. Each buyer is interested in purchasing at most $d_i$ units and has a value $v_i$ per unit. The structure of the matching market is captured by a bipartite graph $G = (B \cup S, F)$ which indicates which buyer is interested in buying goods from which seller.

For example, in the case of internet advertisement, each buyer $b_j \in B$ corresponds to an advertiser and a seller $s_i \in S$ corresponds to a publisher. An edge $(b_j, s_i) \in F$ indicates that buyer $b_j$ is interested in purchasing inventory from the publisher $s_i$'s website. We define $B_i = \{b_j \in B; (b_j, s_i) \in F\}$ as the set of buyers who target seller $s_i$ inventory. Similarly, we define $S_j = \{s_i \in S; (b_j, s_i) \in F\}$ as the set of sellers that are targeted by buyer $b_j$.

We are interested in designing *reservation exchange mechanisms* (or simply reservation mechanisms) which associate for any given matching market described by $(B, S, v, d, \ell, \rho)$ an outcome composed of:

- an allocation $x_i[j] \in \mathbb{Z}$, indicating how many goods from seller $s_i$ are sold to buyer $b_j$, respecting demands $a_j := \sum_i x_i[j] \le d_j$ and supply $c_i := \sum_j x_i[j] \le \ell_i$.
- a total amount $P_j$ paid by each buyer $b_j$, such that the payment per unit doesn't exceed buyer $j$'s value per unit: $P_j \le a_j \cdot v_j$
- a revenue sharing scheme which allocates for each buyer $b_j$ and seller $s_i$, a portion $R_i[j]$ of the buyer's payment $P_j$ to seller $s_i$, such that $\sum_i R_i[j] \le P_j$. We define $R_i := \sum_j R_i[j]$ to be the total revenue obtained by seller $i$.

An outcome satisfying the properties above is said to be a *feasible outcome*. Given a feasible outcome, the utility of involved agents are as follows:

- buyers have quasi-linear utilities, i.e, $u_j = v_j \cdot a_j - P_j$.
- sellers have the revenue minus the reservation price $R_i - \rho_i \cdot c_i$ as their utility.

In the next section, we discuss a set of desirable properties for a reservation mechanism and discuss which subsets of those properties can be simultaneously satisfied.

# 3. AXIOMS FOR RESERVATION EXCHANGE MARKETS

In this section, we develop an axiomatic approach to reservation markets. First, we define a set of desirable properties, referred to axioms, for a well-designed market. As it is the case with axiomatic approaches, some seemingly innocuous axioms might generate impossibility results and some seemingly powerful axioms might not prevent the pitfalls we intended. Here, we define a family of axioms (many with different variations) and discuss, using examples, their strengths and weaknesses.

## 3.1 Fundamental Axioms: efficiency and budget-balance

We establish as our first and most important goal the maximization of market efficiency, which is the sum of the utilities of all agents involved:

- **Efficiency [Eff]:** The implemented outcome maximizes $SW(B \cup S) = \sum_{j \in B} v_j \cdot a_j + \sum_{i \in S} \rho_i \cdot (\ell_i - c_i)$ among all feasible outcomes, assuming the seller derives utility $\rho_i$ for unsold items.

In order to simplify notation, for each seller that has a reserve price $\rho_i > 0$ we add a *proxy buyer* $j(i)$ with demand $d_{j(i)} = \ell_i$ and value $v_{j(i)} = \rho_i$. Let also $j(i)$ be the endpoint of a single edge connecting it to seller $i$. Notice that there is a social-welfare-preserving one-to-one map between outcomes for sellers with reserve prices and sellers without reserves but with proxy buyers. This reduction allows us to ignore from this point on the reserve prices $\rho_i$ and focus on maximizing $\sum_i v_j \cdot a_j$ as the [Eff] goal.

A second goal of the mechanism is to distribute the revenue between sellers. A budget balance mechanism should distribute the entire revenue collected from the buyers to the sellers. A $\beta$-budget balance mechanism should distribute at least a $\beta$-fraction of the revenue.

- **$\beta$-Budget Balance [$\beta$-BB]:** The implemented outcome is $\beta$-budget balance for a constant $\beta \in [0, 1]$ if $\sum_{i \in S} R_i \ge \beta \sum_{j \in B} P_j$. We say that the reservation mechanism is exact budget balance if $\beta = 1$.

## 3.2 Stability properties

A second set of properties describes the stability of the allocation and resilience to manipulation via adding or removing buyers or sellers:

- **Buyer Monotonicity [BM]**: If a new buyer order $b_j$ is added, the revenue of all sellers in $S_j$ does not decrease.
- **Seller monotonicity [SM]**: If a seller increases his reserve price, the revenue of all other sellers does not decrease.

## 3.3 Incentive compatibility

We next define a set of desirable incentives properties for buyers and sellers.

- **[B-IC] Buyer incentive compatibility:** Buyers maximize their utility by reporting their true valuations to the mechanism.
- **[S-IC] Seller incentive compatibility:** Sellers maximize their utility by declaring true reserve prices and supply levels.

Unfortunately, [B-IC] and [S-IC] cannot be simultaneously achieved in a two-sided market [**?**, **?**] if not at the expense of efficiency. Here, we choose to relax Seller incentive compatibility and instead, enforce a fairness constraint among sellers while keeping buyer incentive compatibility.

If we enforce [Eff] and [B-IC], the only mechanism available to decide on the allocation and buyer payments is the VCG mechanism. VCG, however, treats all the sellers as one and therefore doesn't prescribe how the revenue of the auctions should be distributed among the sellers. The central issue in the design of reservation exchange mechanisms is how to distribute the revenue from the VCG auction in a manner that is fair to the sellers. As we will see next, defining a precise notion of fairness that matches our intuition is a quite non-trivial task. First, we show how the most natural definitions fail to capture important situations.

## 3.4 Seller Fairness and Envy-Freeness

We start by identifying a set of properties that we believe a fair revenue sharing scheme should satisfy. The challenge here is in identifying, when a buyer gets some item at price $p$, if a seller can stake a claim on this revenue or not. Firstly, a seller $s_i$ can claim revenue only from buyers that are interested in the inventory owned by seller $s_i$, i.e., $R_i[j] = 0$ for $j \notin B_i$. Also, if a certain buyer $b_j$ never receives goods from seller $s_i$ under any efficient allocation, seller $s_i$ shouldn't be able to claim stake on the revenue $b_j$. This is so because even if this seller drops this connection to the buyer, it won't change the set of efficient outcomes. Moreover, the seller may end up getting a lower revenue because of the reduced competition after dropping such a connection.

In order to capture the above notions, we define for each seller $s_i$, the set $A_i \subseteq B_i$ as the set of buyers that are allocated at least one good from $s_i$ in *some* efficient allocation. We are now able to define the concept of envy-free revenue sharing: roughly speaking, we say that a revenue sharing scheme is envy-free if each seller $s_i$ extracts from $A_i$ more revenue then any other seller with at most the same supply and proportionally more revenue than any dseller with higher supply. More specifically, this concept is defined as follows:

- **Envy-free Revenue Sharing [ERS]**: $\forall s_i, s_{i'}$, $\sum_{j \in A_i} R_i[j] \geq \min(1, \frac{\ell_{s_i}}{\ell_{s_{i'}}}) \cdot \sum_{j \in A_i} R_{i'}[j]$.

If all sellers have the same supply, this boils down to $\sum_{j \in A_i} R_i[j] \geq \sum_{j \in A_i} R_{i'}[j]$. We note that if one is able to design an envy-free mechanism for sellers with unit-supply, this automatically extends to sellers with non-unit supply by the following reduction: transform each seller of supply $\ell_i$ in $\ell_i$ unit supply sellers. An envy free allocation in the transformed setting naturally translates to an envy-free allocation in the original setting. For this reason we will assume for the remainder of the paper that sellers are unit supply.

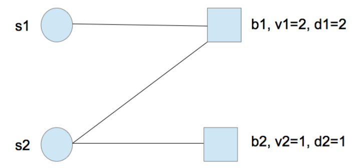


**Figure 1:** The two sellers receive different revenue.

However, as shown in the following example, the above notion of envy-freeness doesn't fully capture the notion of a fair allocation among the sellers:

**Example.** Consider two buyers with valuation $v_1 = 2$, $v_2 = 1$, and demands $d_1 = 2$, $d_2 = 1$. There are two unit supply sellers $s_1, s_2$ with preference constraints shown in Figure **??**. For any mechanism satisfying [Eff], [B-IC] and [ERS], the allocation and payments charged to the buyers must be the one of the VCG mechanism. So, the mechanism sells both items to buyer $b_1$ at total price 1.

Since $A_1 = A_2 = \{b_1\}$, ERS imposes to share the revenue equally between $s_1$ and $s_2$. This way to partition of the revenue can be hardly called fair, since the 1 dollar in revenue is caused by the competition with buyer $b_2$ that is brought to the market by seller $s_2$. So a natural intuition is that a 'fair' scheme should attribute the 1 dollar in revenue to seller 2. ■

The above example implies the need to refine the envy-freeness property to incorporate some notion of *which buyers are responsible for putting the price pressure.* To get a handle on such a notion, we consider ascending auctions to refine our envy-freeness property.

## 3.5 Fairness via ascending auction

The traditional definition of the VCG mechanism is that it allocates according to an efficient allocation and charges each agent according to the externality it imposes on other agents. One problem with this way of defining VCG is that it returns a bundle of items to each agent and a total price but does not specify how much of the payment is attributed to each item. An alternative way to define VCG is via an ascending auction [**?**, **?**], in which there is a price clock $p$ that gradually ascends, and as the price increases items are allocated to buyers. The total payment of the buyer in such a case is the sum over the prices of all individual items, where the price of each item is the value of the price clock when the buyer acquired the item.

The ascending auction description of VCG returns for each buyer $b_j$ his allocation, say $x_j \in \mathbb{Z}_+$ together with $x_j$ prices $p_1 \leq p_2 \leq \ldots \leq p_{x_j}$ corresponding to the value of the price clock when he acquires each of those items. In other words, we can describe the outcome of VCG as an ascending auction as a set of $n$ *buying events*, where $n = \sum_i \ell_i$ and each buying event is a pair $(b_j, p)$ indicating that one item was sold to buyer $b_j$ at price $p$. Note that we assume all items are sold by VCG, which is always the case when we consider proxy buyers as discussed in Section **??**.

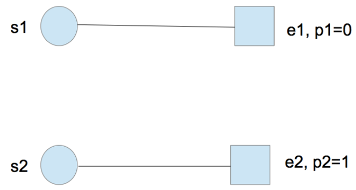


**Figure 2:** Each seller is linked in the clinching graph to only one buying event.

Let $E$ be the set of buying events that represents the outcome of the auction. During its execution, the ascending auction maintains in each step a tentative assignment of buying events to sellers. This allows us to define a bipartite graph between sellers and buying events called the *clinching graph*. We say that a seller $s_i$ is connected to buying event $e_j$ if this seller is tentatively allocated to that buying event in some point of the auction execution. If the auction execution is not unique (because of ties, caused for example by two identical sellers) we consider an edge to be in the clinching graph if for some execution of the auction its corresponding buying event is connected to the corresponding seller.

We postpone a formal definition of the ascending auction until Section **??**, but we illustrate its execution for the instance in Example **??**. The price clock starts at zero, and at that price the auction is already able to allocate the item in $s_1$ to buyer $b_1$, since he faces no competition on that item. This generates a buying event $(b_1, 0)$ that is associated with seller $s_1$. For prices between 0 and 1, both buyers compete for the remaining item. When the price clock reaches 1, buyer $b_2$ is no longer interested in the remaining item and buyer $b_1$ can acquire it at price 1, generating a buying event $(b_1, 1)$, which is associated with seller $s_2$. There are no ties, so this is the unique execution of the auction, what generates the clinching graph represented in Figure **??**.

Given the clinching graph, we are now able to define a stronger notion of envy-freeness based on it. We denote by $r_i[j]$ the revenue obtained by seller $s_i$ from buying event $e_j \in E_i$. Let us denote the revenue of unit supply seller $s_i$ by $r_i = \sum_{j \in E_i} r_i[j]$. We now state a new version of Envy-free Revenue Sharing, that we denote by ERSCG, as follows:

- **Envy-free Revenue Sharing from Clinching Graph [ERSCG]**: $\forall s_i, s_{i'}, \sum_{j \in E_i} r_i[j] \geq \sum_{j \in E_i} r_{i'}[j]$.

According to the definition of clinching graph, we obtain for the example of Figure **??** that $E_1 = \{e_1\}$ and $E_2 = \{e_2\}$. The revenue sharing scheme that attributes $r_1[1] = 0$, $r_1[2] = 0$, $r_2[1] = 0$ and $r_2[2] = 1$ is therefore ERSCG. We conclude that the new definition of envy-freeness is able to characterize a fair sharing of the revenue between sellers.

We also define an approximate version of the previous property:

- **$\alpha$-Envy-free Revenue Sharing from Clinching Graph[$\alpha$-ERSCG]:** $\forall s_i, s_{i'}, \sum_{j \in E_i} r_i[j] \geq \alpha \sum_{j \in E_i} r_{i'}[j]$.

We conclude by observing that the notion of envy-freeness we introduce can easily be adapted to the original non-unit supply sellers.

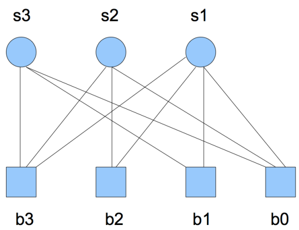


**Figure 3:** The buyer-seller graph for the counter example to envy-freeness

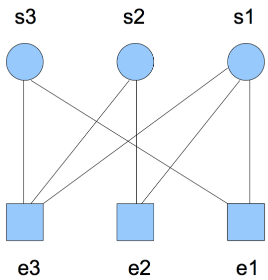


**Figure 4:** The clinching graph for the counter example to envy-freeness

## 4. IMPOSSIBILITY OF ENVY-FREENESS AND BUDGET BALANCE

Before presenting two revenue sharing schemes based on the definition of clinching graph in Sections **??** and **??**, we show that envy-freeness and budget balance are indeed contradicting objectives for any revenue sharing scheme based on an efficient allocation.

THEOREM 4.1. *There does not exist any revenue sharing efficient mechanism for the reservation exchange market which is BB and $\alpha$-ERSCG for $\alpha \geq \sqrt{3} - 1 + \delta \approx 0.732$, for an arbitrary small value $\delta > 0$.*

PROOF. The proof is based on the following example. We have four buyers and three unit-supply sellers. Buyer $b_0$ has demand $d_0 = 3$ and valuation $v_0 = \epsilon > 0$, buyer $b_1$ has demand $d_1 = 2$ and valuation $v_1 = 1$ and buyers $b_2$ and $b_3$ have demand $d_2 = d_3 = 1$ and valuation $v_2 = v_3 = 2$. The buyer-seller graph is shown in Figure **??**.

The three buyers receive one item each. The corresponding clinching graph is shown in Figure **??**. We have payments $p_1 = \epsilon, p_2 = 1, p_3 = 1$.

Denote by $x_i[j]$ the share of $p_j$ attributed to seller $s_i$. If the revenue sharing scheme is exact budget balance then we have $\sum_{j \in E_i} x_i[j] = 1, \forall s_i \in S$, and $\sum_{s_i : e_j \in E_i} x_i[j] = 1, \forall e_j \in E$.

Assume in the example $x_1[1] = x$ and $x_3[1] = 1 - x$. From exact budget balance we have $x_3[3] = x$, $x_2[3] + x_1[3] = 1 - x$ and $x_1[2] + x_1[3] = 1 - x$.

Let us denote by $\alpha$ the maximum ERSCG ratio that can be achieved. Let us state $\alpha$-ERSCG of $s_1$ with respect to $s_2$:

$$\sum_{e_j \in E_1} r_1[j] \geq \alpha \sum_{e_j \in E_1} r_2[j].$$

We obtain

$$\epsilon x + (1 - x) \geq \alpha.$$

Next, we state $\alpha$-ERSCG of $s_3$ with respect to $s_1$ and $s_2$:

$$\sum_{e_j \in E_3} r_3[j] \geq \alpha \ \mathtt{max} \, \{ \sum_{e_j \in E_3} r_2[j], \sum_{e_j \in E_3} r_1[j] \}.$$

Given that the revenue of at least one of $s_2$ and $s_1$ on $e_3$ is at least $(1-x)/2$, we obtain

$$(1 - x)\epsilon + x \geq \alpha \frac{1-x}{2}.$$

We therefore conclude that $\alpha \leq \min\{1 - x + \epsilon x, 2\epsilon + \frac{2x}{1-x}\}$. By maximizing $\alpha$ as function of $x$, and by setting $\epsilon$ arbitrarily small, we obtain $\alpha \leq \sqrt{3} - 1 + \delta$ for an arbitrary small $\delta > 0$.

□

# 5. REVENUE SHARING BY THE CLINCHING ASCENDING AUCTION.

The first revenue sharing scheme is based on the allocations computed by the clinching ascending auction (CA). We denote by $CA$ this revenue sharing scheme.

A detailed description of the use of the clinching ascending auction [**?**, **?**] to compute efficient VCG allocations is in Appendix. We specifically present a version for matching markets given in [**?**].

Crucial to the definition of revenue sharing scheme is the notion of *priority order* among sellers that is used in the execution of the CA. Whenever the CA is indifferent about buying between a set of sellers, it decides in "favor" of the seller with lower priority in the precedence order. Intuitively, the seller of higher priority will enjoy a payment that is at least as good as the lower priority seller since the price in the ascending auction is non-decreasing. A priority order between sellers is simply represented by a permutation $\pi \in \Pi(S)$ where $\Pi(S)$ defines the set of all permutations of set $S$.

We set $r_i^\pi[j] = p_j$ if the execution of CA on permutation $\pi$ matches $e_j$ to $s_i$.

Revenue share of seller $s_i$ from buying event $e_j$ is defined as

$$r_i[j] = \mathbb{E}_{\pi \in \Pi(S)}[r_i^\pi[j]]. \quad (1)$$

The revenue of seller $s_i$ is defined as $r_i = \sum_{e_j \in E_i} r_i[j]$. Since the total revenue of the mechanism $REV = \sum_{j \in E} p_j$ is shared between sellers, we state a first property of the revenue sharing scheme CA:

CLAIM 5.1. *The revenue sharing scheme CA is BB.*

We next prove that the revenue sharing scheme CA is not exact ERSCG.

THEOREM 5.2. *The revenue sharing scheme CA is at most* 1/2*-ERSCG.*

PROOF. Consider the example of Figure **??**.

We have three buyers and three unit supply sellers. The three buyers are $b_1$ with $d_1 = 2$ and valuation $v_1 = 1$, $b_2$ and $b_3$ with demand $d_2 = d_3 = 1$ and valuation $v_2 = v_3 = 2$. Figure **??** shows the preference sets of the buyers that also

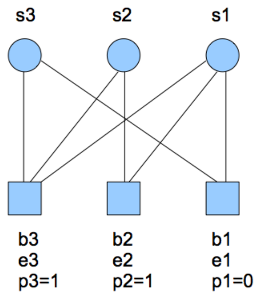


**Figure 5:** The clinching graph for the counter example to ERSCG

correspond with the edges of the clinching graph. The three buyers receive one item each. We therefore have buying events $\{e_1, e_2, e_3\}$ with payments $p_1 = 0$, $p_2 = p_3 = 1$. Let us compute the revenue shares of $s_1$ and $s_2$. Seller $s_1$ is matched to $e_1$ on permutations $123, 132, 213$, $e_2$ on permutation $312$, $e_3$ on permutations $321, 231$. Seller $s_2$ is always matched to $e_2$ or $e_3$. The mechanism is therefore not envy-free since the revenue from $e_1$ is 0. In particular we observe that $s_1$ achieves half the revenue of $s_2$ . □

We next prove the approximate envy-freeness of revenue sharing scheme CA. The proof is given in appendix.

THEOREM 5.3. *The revenue sharing scheme CA is* 1/2*-ERSCG.*

We conclude with the properties of buyer monotonicity and seller monotonicity for revenue sharing scheme CA. The proofs are deferred to the Appendix.

THEOREM 5.4. *BM and SM hold for revenue sharing scheme CA.*

# 6. REVENUE SHARING BY THE EATING MECHANISM.

The eating mechanism is defined as a fractional process in time on the clinching graph $CG = (E \cup S, H)$, $H = \{(e_j, s_i) : e_j \in E_i\}$. At each time $x \in [0, 1]$, the unit supply seller $s_i$ "eats" from the the non-exhausted buying event $e_j \in E_i$ of highest payment $p_j$. Each seller will eat at most up to a fraction of 1. A buying event is exhausted when it has been eaten for 1 unit. The result of the eating mechanism is a fractional assignment $x_i[j] \in [0, 1]$ such that $\sum_{j \in E_i} x_i[j] \leq 1$ for each seller $s_i \in S$ and $\sum_{s_i : e_j \in E_i} x_i[j] \leq 1$ for each buying event $e_j \in E$.

Revenue share of seller $s_i$ from buying event $e_j$ is defined as

$$r_i[j] = x_i[j] \times p_j. \quad (2)$$

The total revenue of seller $s_i$ is also equal to $r_i = \sum_{j \in E_i} r_i[j]$.

It is easy to observe that the revenue shares by the eating mechanism can be computed in polynomial time. We first show that revenue share mechanism EM is envy-free.

THEOREM 6.1. *The revenue sharing scheme EM is ERSCG.*

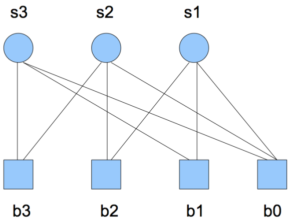


**Figure 6:** The example for the eating mechanism.

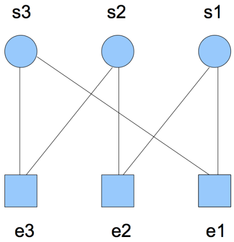


**Figure 7:** Eating mechanism is not exact budget balance.

It is not clear that in the EM sharing scheme all sellers eat up to 1. We show in the following that EM is not exact budget balance.

THEOREM 6.2. *For any $\epsilon > 0$, the revenue sharing scheme EM is at most $\frac{11+\epsilon}{12+\epsilon}$-BB.*

PROOF. In the example of Figure **??**, buyer $b_0$ has demand $d_0 = 3$ and valuation $v_1 = 1$, buyer $b_1$ has demand $d_1 = 2$ and $v_1 = 1 + \epsilon$, and finally buyers $b_2$ and $b_3$ have $d_2 = d_3 = 1$ and valuation $v_2 = v_3 = 2$. There are three sellers. The buyer/seller graph is shown in Figure **??**.

The corresponding clinching graph is shown in Figure **??**.

The three buyers receive one item each. Buying event $e_1$ happens at price 1, buying events $e_2$ and $e_3$ happen at price $1 + \epsilon$. It is clear that the three sellers will eat first from $e_2$ and $e_3$ and later from $e_1$. Now the issue is whether seller $s_2$ prefers $e_3$ to $e_2$ or $e_2$ to $e_3$.

If seller $s_2$ prefers $e_3$ to $e_2$, seller $s_3$ eats 1/2 of $e_3$ and 1/2 of $e_1$, seller $s_2$ eats 1/2 of $e_3$ and 1/4 of $e_2$, seller $s_1$ eats 3/4 of $e_2$ and 1/4 of $e_1$. If seller $s_2$ prefers $e_2$ to $e_3$, seller $s_3$ eats 3/4 of $e_3$ and 1/4 of $e_1$, seller $e_2$ eats 1/2 of $e_2$ and 1/4 of $e_3$, seller $s_1$ eats 1/2 of $e_2$ and 1/2 of $e_1$. In both cases $s2$ eats up to 3/4 and $e_1$ is eaten up to 3/4. We conclude that a revenue of 1/4 out of total revenue of $3 + 2 \times \epsilon$ is not distributed between sellers. □

The proof of the main theorem of this section is given in Appendix.

THEOREM 6.3. *The revenue sharing scheme EM is $1 - 1/e$-BB.*

# 7. CONCLUSIONS

The reservation exchange market is an emerging model for internet advertising that brings together multiple publishers and advertisers interested in trading inventories of impressions available in the future. In this work, we present the axioms and the design principles at the basis of mechanisms for reservation exchange markets. The goal we define for these markets is the design of mechanisms that are incentive compatible for buyers, envy-free for sellers, efficient and budget balance. We show that this is possible if one of the requirements of budget balance or envy-freeness is slightly relaxed. Our efficient revenue sharing mechanisms are based on the notion of clinching graph that is a convenient representation of the trades of efficient VCG allocations.

We leave several open problems in the context of reservation exchange markets. First of all, it would be interesting to close some of the gaps on approximate envy-freeness and budget balance of the revenue sharing mechanisms we propose. It is also unknown whether the eating mechanism holds some of the monotonicity properties we define in this paper. Moreover, since the clinching graph seems to provide fundamental insights for designing fair revenue sharing mechanisms, it would be helpful to derive its structure from basic properties of VCG mechanisms.

**Acknowledgements.** We thank Marek Adamczyk for suggesting the analysis of budget balance of the eating mechanism.

# APPENDIX

For lack of space, some of the proofs are omitted.

# A. CLINCHING ASCENDING AUCTION

In this section, we show how to use the clinching ascending auction (CA) [**?**, **?**, **?**, **?**] to compute efficient VCG allocations.

In CA, we will use a notion of *priority order* among sellers. A priority order between sellers is simply represented by a permutation $\pi \in \Pi(S)$ where $\Pi(S)$ defines the set of all permutations of set $S$. The ascending auction raises the price of unsold supply till all items are sold.

The auction makes an extensive use of maximal $D$-matchings. A $D$-matching in the graph $G = (B \cup S, F)$ is a subgraph of $G$ where vertices have degree constraints. Given the vector $D$ of degree constraints, and a graph $G = (B \cup S, F)$, one can compute a maximal weight $D$-matching in polynomial time. We denote such $D$-matchings simply by *matchings*. In our case, the bipartite graph is $G = (B \cup S, F)$ has a vector of degree constraints given by the residual demand for buyers and by 1 for the unit supply sellers.

The main details of the auction are described by Algorithms **??**, **??** and **??**.

Throughout the execution of the auction there is always some current price $p$ (initially zero); a set $S$ of unsold sellers of cardinality $n$, and current remaining demand $d_j$ from buyer $b_j$. We assume the existence of a maximum matching of size $n$ in the graph thus guaranteeing that all items will be sold in the course of the auction.

The auction repeatedly computes a matching of unsold items, i.e., a maximal matching that assigns at most $d_j$ out of $n$ unsold items to buyer $b_j$. More precisely, given a permutation $\pi$, CA repeatedly computes $(X, \pi)$-avoid matchings. These are maximal matchings that try to avoid, if at all possible, assigning items to buyers in some set $X$. Moreover, among all such matchings, it is returned the one that is lexicographically minimum with respect to seller order $\pi$.

**Algorithm 1** Clinching Auction

**procedure** CLINCHING AUCTION($B, d, S, \pi, \{S_j\}, \{B_i\}$)
Implicitly defined $A$, $V$ and $n$ — see Equations (**??**) and (**??**).
$D(\neg\{b_j\}, \pi)$ - number of items assigned to agents in $A \setminus \{b_j\}$ in $(\{b_j\}, \pi)$-avoid matching
  $p \leftarrow 0$
  **while** $(A \neq \emptyset)$ **do**
    Sell($V$)
    $A \leftarrow A - V$
    **repeat**
      **if** $\exists b_j | D(\neg\{b_j\}, \pi) < n$ **then** Sell($b_j, \pi$)
      **end if**
    **until** $n = 0$ or $\forall i$: $D(\neg\{b_j\}, \pi) = n$)
    If $n > 0$, increase $p$ until for some $b_j \in A$, $p = v_j$
  **end while**
**end procedure**

**Algorithm 2** Computing an avoid matching, can be done via min cost max flow

**procedure** ($X$,$\pi$)-AVOID MATCHING
Construct interest graph $G$:

- Active buyers, $A$, on left, demand of buyer $b_j \in A$ $= d_j$
- Sellers $S$, on the right, supply of seller $s_i \in S$, $l_i = 1$.
- Edge $(b_j, s_i)$ from buyer $b_j \in A$ to seller $s_i \in S$ iff $s_i \in S_j$ and $b_j \in B_i$.

Return maximum $D$-matching with minimum number of items assigned to buyers of $X$ that is lexicographically maximal on $\pi$.
**end procedure**

**Algorithm 3** Selling to a set $X$ of Buyers

**procedure** SELL($X, \pi$) Sell to buyers $X$ according to $\pi$.
  **repeat**
    Compute $Y = (X$,$\pi)$-AVOID MATCHING
    For $s_i$ of lowest priority such that $(b_j, s_i) \in Y$ and $b_j \in X$, sell seller $s_i$ to buyer $b_j$. Update demand $d_j = d_j - 1$.
  **until** $D(\neg X, \pi) \geq n$
**end procedure**

The mechanism will sell items only when the $(S, \pi)$-avoid matching will still assign items to buyers of $S$. Whenever buyers in $X$ can be matched to different sellers, given the lexicographically minimum order, the precedence is given to sellers that have lower priority in $\pi$.

The algorithm also keeps a set of active buyers $A$ — those with current demand greater than zero and valuation $v_j \geq p$. Not all active agents are in the same position with respect to the auction. The auction will distinguish a set of buyers $V$ with valuation equal to the current price:

$$A = \{1 \leq j \leq n | d_j > 0\}, \quad (3)$$
$$V = \{1 \leq j \leq n | d_j > 0, v_j = p\}. \quad (4)$$

The algorithm repeatedly tries to sell items at the current level of demand for all the buyers. When no items can be sold the price increases. In algorithm **??**, we compute matchings that obey the following three requirements: i) maximize the number of sellers that are matched; ii) minimize the number of sellers matched to buyers of $X$; iii) lexicographically maximal on $\pi$. The three requirements are listed in the order of priority, i.e., firstly maximize number of matched sellers, secondly minimize number of buyers of $X$ that are matched, and thirdly, match sellers in lexicographically maximal order according to $\pi$. The last requirement implies that there exists no other matching satisfying i) and ii) that matches a seller of higher priority in place of a seller of lower priority to a buyer of $X$.

The auction will in fact sell all items. Once a price has been updated, the auction checks if it must sell items to value limited buyers. Such buyers will receive no real benefit from the items (their valuation is equal to their payment), but this is important so as to increase the utility of the auctioneer and the profit of the sellers. To check if this is necessary, the auction computes a $(V, \pi)$-avoid matching, trying to avoid the bidders in $V$. If this cannot be done, then items are sold to these $V$ bidders according to the order of $\pi$. After items are sold to value limited bidders, these bidders effectively disappear and we are only left to consider selling to active bidders.

The main loop of the mechanism checks whether any items must be sold to any of the currently active bidders. The auction sells an item to some buyer, $b_j$, where the total demand of remaining bidders is such that an item can be assigned to $b_j$ without creating a shortage. This makes use of the $(\{b_j\}, \pi)$-avoid matching, if in the $(\{b_j\}, \pi)$-avoid matching some item is matched to $b_j$ then $b_j$ must be sold that item. The seller is decided according to the precedence order $\pi$. If no item can be sold the price increases.

# B. PROOF OF THEOREM 5.2

PROOF. We show that for every two sellers $s_i, s_{i'}$, for the revenues computed of revenue sharing scheme CA, it holds: $\sum_{j \in E_i} r_i[j] \geq 1/2 \sum_{j \in E_i} r_{i'}[j]$. We conclude that revenue sharing scheme CA is 1/2-ERSCG.

Let us denote by $e_1, \ldots, e_n$ the $n$ buying events and by $p_1, \ldots, p_n$ the corresponding payments. Buying events are ordered by non-increasing prices $p_1, \ldots, p_n$. It does not depend on the precedence $\pi$ between sellers.

We also denote by $j^\pi(i) \in E_i$ the index of the buying event $e_{j^\pi(i)}$ sold to seller $s_i$ in permutation $\pi$. We partition set $\Pi(S)$ into three sets:

1. $\Pi^1(S) = \{\pi \in \Pi(S) : \{j^\pi(i), j^\pi(i')\} \in E_i \cap E_{i'}\}$

2. $\Pi^2(S) = \{\pi \in \Pi(S) : j^\pi(i') \notin E_i\}$

3. $\Pi^3(S) = \{\pi \in \Pi(S) : j^\pi(i') \in E_i, j^\pi(i) \notin E_{i'}\}$

LEMMA B.1. *For the permutations of $\Pi^1(S)$, it holds*

$$\mathbb{E}_{\pi\in\Pi^1(S)}[r_i^\pi[j^\pi(i)]] = \mathbb{E}_{\pi\in\Pi^1(S)}[r_{i'}^\pi[j^\pi(i')]]$$

We now consider permutations from $\Pi^2(S)$.

LEMMA B.2. *For the permutations of $\Pi^2(S)$, it holds*

$$\mathbb{E}_{\pi\in\Pi^2(S):j^\pi(i')\in E_i}[r_{i'}^\pi[j^\pi(i')]] = 0 \tag{5}$$

We finally consider permutations $\Pi^3(S) = \{\pi \in \Pi(S) : j^\pi(i') \in E_i, j^\pi(i) \notin E_{i'}\}$.

LEMMA B.3. *For the permutations of $\Pi^3(S)$, it holds*

$$\mathbb{E}_{\pi\in\Pi^3(S)}[r_i^\pi[j^\pi(i)]] \geq 1/2\mathbb{E}_{\pi\in\Pi^3(S)}[r_{i'}^\pi[j^\pi(i')]]$$

From the three above lemmas, it follows:

$$\begin{aligned}
\sum_{j\in E_i} r_i[j] &= \mathbb{E}_{\pi\in\Pi(S)}[r_i^\pi[j^\pi(i)]] \\
&= \mathbb{E}_{\pi\in\Pi^1(S)}[r_i^\pi[j^\pi(i)]] + \mathbb{E}_{\pi\in\Pi^2(S)}[r_i^\pi[j^\pi(i)]] \\
&+ \mathbb{E}_{\pi\in\Pi^3(S)}[r_i^\pi[j^\pi(i)]] \\
&\geq \mathbb{E}_{\pi\in\Pi^1(S)}[r_{i'}^\pi[j^\pi(i')]] \\
&+ \mathbb{E}_{\pi\in\Pi^2(S):j^\pi(i')\in E_i}[r_{i'}^\pi[j^\pi(i')]] \\
&+ \frac{1}{2}\mathbb{E}_{\pi\in\Pi^3(S)}[r_{i'}^\pi[j^\pi(i')]] \\
&\geq \frac{1}{2}\sum_{j\in E_i} r_{i'}[j],
\end{aligned}$$

thus proving the claim.

□

# C. PROOF OF THEOREM 5.4

Revenue share of seller $s_i$ from buying event $e_j$ in the revenue sharing schemes CA is defined as

$$r_i[j] = \mathbb{E}_{\pi\in\Pi(S)}[r_i^\pi[j]], \tag{6}$$

with $r_i[j]$ defined as the revenue of seller $s_i$ on buying event $e_j$ in the execution of CA on permutation $\pi$. Denote by $m$ the number of buyers, i.e., $m = |B|$. We show properties BM and SM as a corollary of a general monotone property of of the CA executed on permutation $\pi$ when the valuation of buyer $b_m$ is increased from $v_m = p$ to $v_m = p+\epsilon$, for an arbitrary $\epsilon > 0$. W.l.o.g, assume that $b_m$ is the buyer of higher priority in the execution of the clinching auction, i.e., whenever the auction has the chance to sell at price $p$ to more that one buyers, it does it according to a fixed priority. Denote by $d_j(p)$ the residual demand of buyer $b_j$ after the CA has been executed till price $p$, $p$ included. Denote by $S(p,\pi)$ the set of sellers unsold after the CA has been executed till price $p$ on permutation $\pi$. Denote by $X'$ the quantity $X$ when buyer $b_m$ has valuation $v_m = v + \epsilon$.

Our revenue monotonicity properties stem from the following general claim:

CLAIM C.1. *For any price $p \geq 0$, permutation $\pi$ and buyer $b_j$ it holds:*

*i.* $d_j(p) \leq d'_j(p)$;

*ii.* $S(p,\pi) \subseteq S'(p,\pi)$.

THEOREM C.2. *BM holds for revenue sharing scheme CA.*

PROOF. The property states that the revenue of all sellers does not decrease if a new buyer is added. We can simulate the addition of a new buyer $b_m$ by initially setting $v_m = 0$ and then increasing $v_m$ till the exact valuation of the buyer. From the monotonicity property of Lemma **??**, in particular from $S(p,\pi) \subseteq S'(p,\pi)$, we conclude that all sellers will not be sold at a lower price if a new buyer is inserted. We therefore conclude with the proof of the theorem. □

Similarly, we prove:

THEOREM C.3. *SM holds for revenue sharing scheme CA.*

# D. PROOF OF THEOREM 6.3

PROOF. Let $H$ be the set of arcs between sellers $S$ and events $E$. Let $x(t) \in [0,1]^H$ represent process of eating over time for $t \in [0,1]$. Let $H_e$ be arcs adjacent to an eating event $e$. Consider function $f(x(t)) = \sum_{e\in E} p_e \cdot \left(\sum_{a\in H_e} x_a(t)\right)$. Events have prices $p_1 \geq ... \geq p_n$, and suppose there exists a perfect matching so that $OPT = p_1+...+p_n$ (this is without loss, reasoning for just a maximum matching is the same).

The following Lemma holds.

LEMMA D.1. *At any time $\tau \in [0,1]$ it holds that*

$$\frac{d^+ f(x(t))}{dt}(\tau) + f(x(\tau)) \geq OPT.$$

Given the Lemma we have that

$$\frac{d^+\left(e^t f(x(t))\right)}{dt}(\tau) = e^\tau \frac{d^+ f(x(t))}{dt}(\tau) + e^\tau f(x(\tau)) \geq e^\tau OPT.$$

Let $T_1 \leq T_2 \leq ... \leq T_n$ be moments in which events where becoming exhausted; add $T_0 = 0$ and $T_{n+1} = 1$ for simplicity. In the intervals $[T_{i-1}, T_i]$ the function $f$ is differentiable, and the derivative of $f(x(t))$ is equal to its right-side derivative, so

$$\begin{aligned}
&e^{T_i} f(x(T_i)) - e^{T_{i-1}} f(x(T_{i-1})) \\
=& \int_{T_{i-1}}^{T_i} \frac{df(x(t))}{dt}(\tau)\ d\tau \geq \int_{T_{i-1}}^{T_i} e^\tau \cdot OPT\ d\tau \\
=& \left(e^{T_i} - e^{T_{i-1}}\right) OPT.
\end{aligned}$$

Summing for $i$ from 1 to $n+1$ gives us

$$\begin{aligned}
&e \cdot f(x(1)) - f(x(0)) \\
=& \sum_{i=1}^{n+1} e^{T_i} f(x(T_i)) - e^{T_{i-1}} f(x(\tau_{i-1})) \\
\geq& \sum_{i=1}^{n+1} \left(e^{T_i} - e^{T_{i-1}}\right) OPT \\
=& (e-1) OPT.
\end{aligned}$$

Given that $f(x(0)) = 0$ we get $f(x(1)) \geq \left(1 - \frac{1}{e}\right) OPT$. Remains to notice that $f(x(1))$ is our solution. □

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