Bernoulli Factories

Renato Paes Leme
(Google Research)

(Bernoulli wearing a factory hat)

Bernoulli variable = p-coin

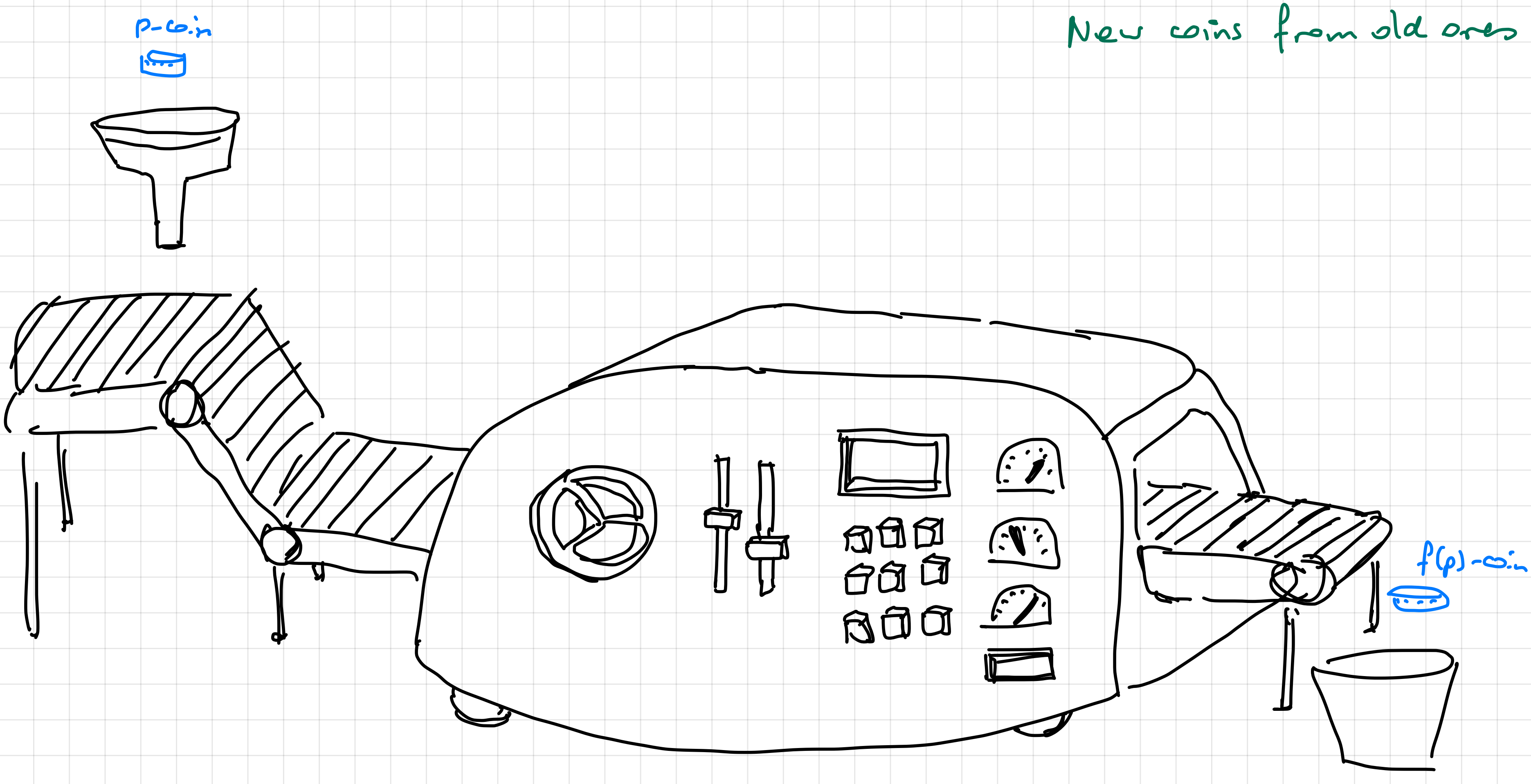
$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$\boxed{\text{unknown } p}$

www.renatoppl.com/bernoulli

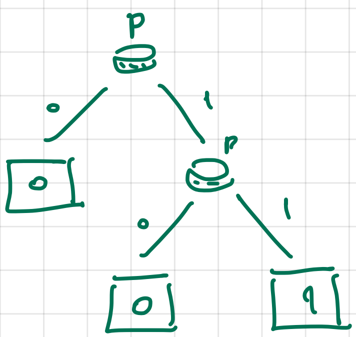
Bernoulli Factories

New coins from old ones

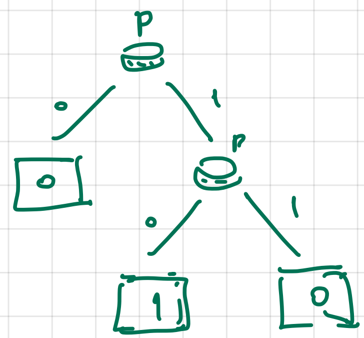


Examples of Simple Factories

① $f(p) = p^2$



② $f(p) = p(1-p)$



③ Bernstein monomials: $p^a (1-p)^b$ $a, b \in \mathbb{Z}_+$

Sample $a+b$ flips

Output 1 if $\underbrace{11\,1..1}_{a}\ \underbrace{0\ldots0}_{b}$

Single Parameter Bernoulli Factories

Given a function $f : S \subseteq (0,1) \to [0,1]$,

design an algorithm (decision tree)

that samples an $f(p)$-coin from a p-coin

(of unknown bias).

[Keane O'Brien 1994]

Exact Sampling

$\downarrow$

Simulation

mech design

$\leftarrow$ Incentive Compatible

Sample k times

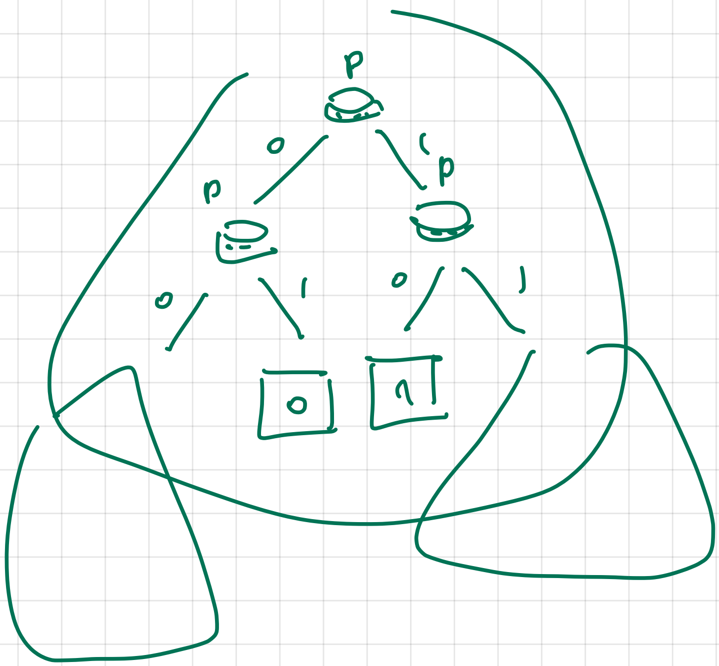
$$\hat{p} = \frac{\sum_{i=1}^{k} X_i}{k}$$

$$f(\hat{p})$$

Von Neumann's Procedure

Sample an unbiased coin from a biased one

(4) $f(p) = \frac{1}{2}$ for $p \in (0,1)$



Von Neumann's Procedure

Assume $\overset{x}{\text{coin}}$ for all x

Sample a coin of bias x from a biased one

(5) $f(p) = x$ for some fixed $x \in [0,1)$

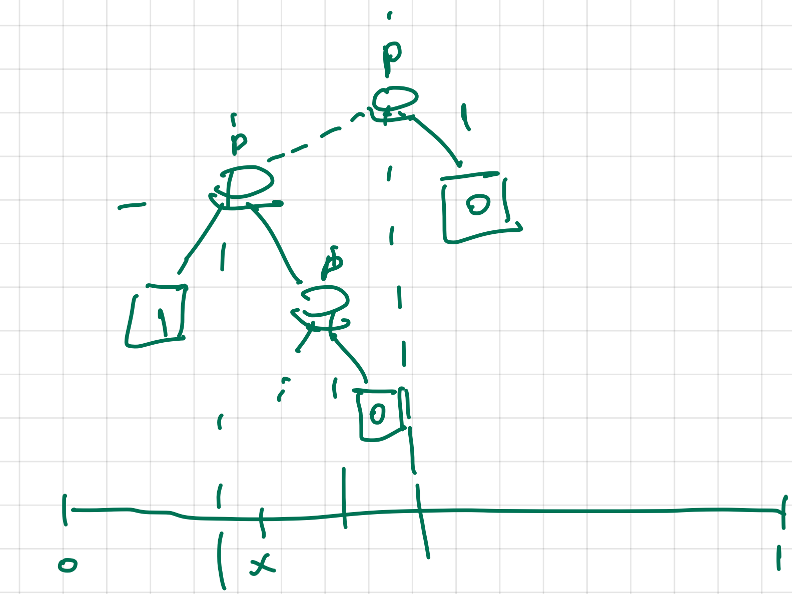
$p \Rightarrow 1/2$

Sample $u \sim \text{Uniform}[0,1)$

Output 1 if $u \leq x$.

$$u = \sum_{i=1}^{\infty} \frac{1}{2^i} u_i$$

binary digits of u.



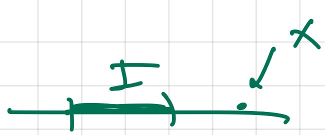
$I = [0,1)$

while $x \in I$

Sample $\overset{1/2}{\text{coin}}$

if $\boxed{0}$ $I = [a, \frac{a+b}{2})$

if $\boxed{1}$ $I = [\frac{a+b}{2}, b)$



if $I < x$ output $\boxed{0}$

if $I > x$ output $\boxed{1}$

Bernstein Polynomials

6 Bernstein polynomials $f(p) = \sum_{i=1}^{k} c_i \, p^{a_i} (1-p)^{b_i}$

$a_i, b_i \in \mathbb{Z}_+$

$c_i \geq 0$

$\sum c_i \leq 1.$

Sample index i with prob c_i

sample p $(a_i + b_i)$-times

7 $f(p) = \dfrac{p}{2-p} = \sum_{k=1}^{\infty} \dfrac{1}{2^k} p^k$

Sample index k w.p. $1/2^k$

Sample $k \times$ p

If all 1's output $\boxed{1}$

Otherwise output $\boxed{0}$.

$f(p) = \mathbb{E}[p^X]$

$X \sim \text{Geometric}(1/2).$

Other Examples

(8) Moment Generating Functions, e.g. $f(p) = e^{p-1}$

$f(p) = E[p^X]$ random variable X on $\mathbb{Z}_+$.

$f(p) = e^{p-1} = E[p^X]$ $X \sim \text{Poisson}(1)$.

$$e^{p-1} = \sum_{k=0}^{\infty} \underbrace{\frac{1}{e}\frac{1}{k!}}_{\text{prob distribut}} p^k$$

Complete Characterization

Which functions $f : S \subseteq (0,1) \longrightarrow [0,1]$

admit Bernoulli factories ?

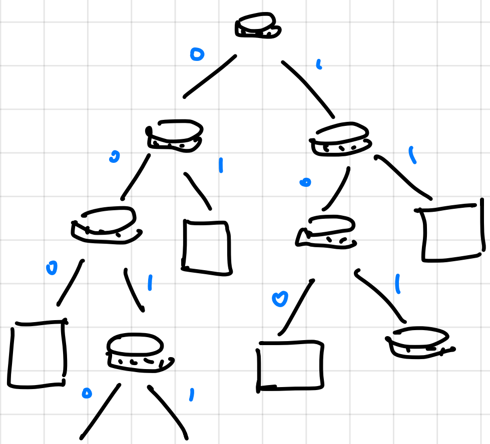
1. What is a general factory?

2. Examples of functions that don't admit factories

Generic Factory

$$f(p) = \sum c_i \, p^{a_i} (1-p)^{b_i}$$

Bernstein polynomials



Factory = Decision Tree

Nodes:
- p (p-coin)
- c (helper coin)
- $\boxed{0}$ $\boxed{1}$ (output)

- Finite vs Infinite:

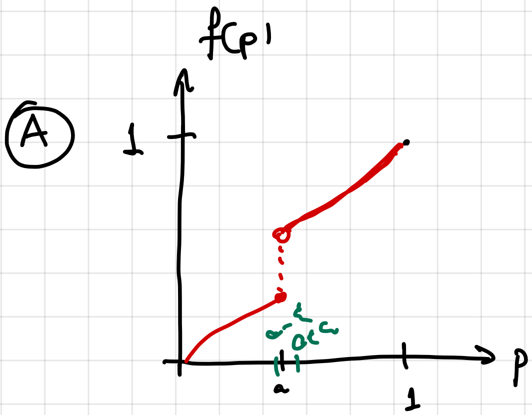
- Termination: $\mathbb{P}(\mathcal{F}(p) = \emptyset) = 0.$

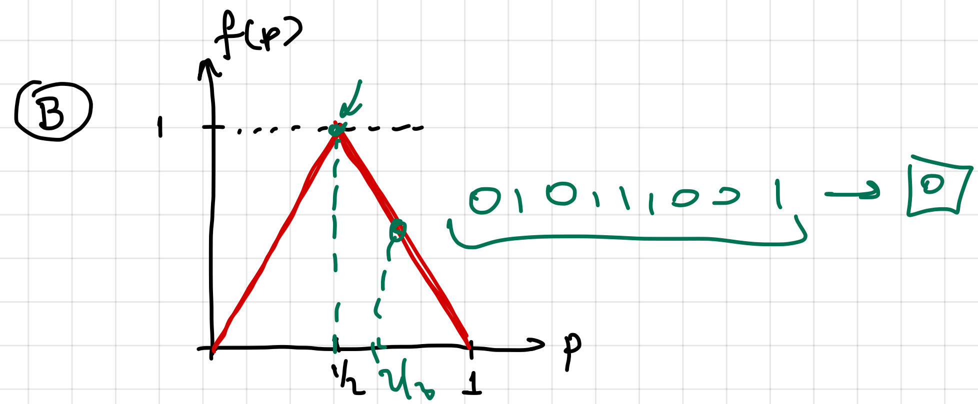
  $\mathcal{F}(p)$ random-variable parametrized by p.

  $\mathcal{F}(p) \in \{0, 1, \emptyset\}$

- $\mathbb{P}[\mathcal{F}(p) = 1] = f(p)$

Functions Without Factories





iid samples from p

$[0100011100011100 \ldots]$

$p = 1/2$

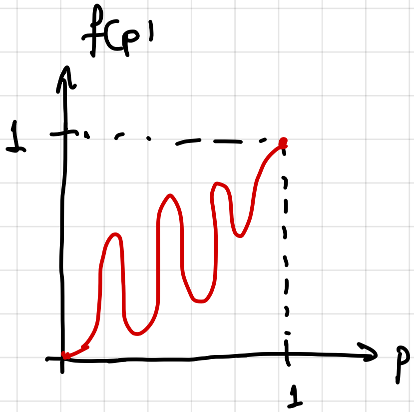
$f(p)$ must be continuous to admit a factory.

If $f(p) = 1$ for some $p \in (0,1)$ then $f(p) \equiv 1$ is the only factory you can implement.

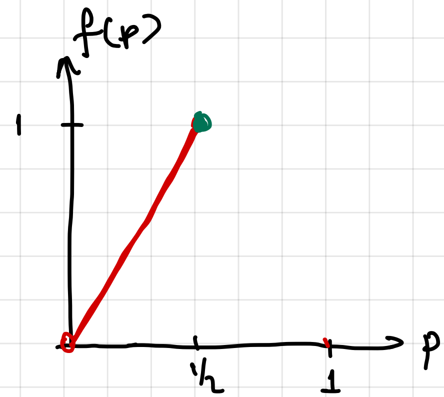
Functions Without Factories

Test Your Intuition. Which of these functions has a factory?

Ⓒ ✓

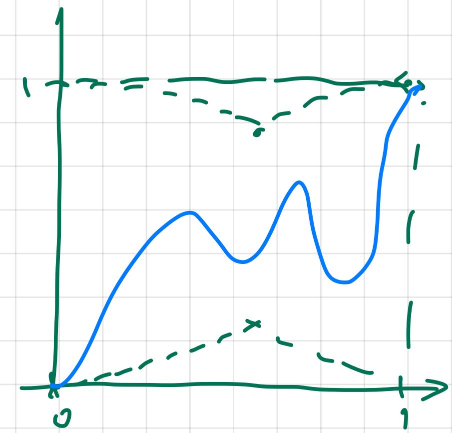


Ⓓ ✗



$f: (0, 1/2) \to (0, 1)$

Theorem (Keane, O'Brien)



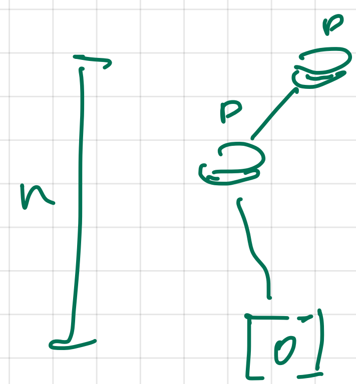
Function $f: S \subseteq (0,1) \to (0,1)$ admits a factory if:

(1) it is continuous

(2) f is constant or poly-bounded

$\exists n$ s.t. $\min\left[f(p), 1-f(p)\right] \geq \min\left(p^n, (1-p)^n\right)$

Proof: Necessity: (1) ✓ (2) ✓



$$1 - f(p) \geq p^a (1-p)^b \qquad a+b=n$$

$$\geq \min(p, 1-p)^{a+b}$$

Theorem (Keane, O'Brien)

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$\exists n$ s.t. $\min[f(p), 1-f(p)] \geq \min(p^n, 1-p^n)$

<u>Lemma</u>: Given f as in Thm, there is a function g implementable by a finite factory s.t.

$$0 \leq f(p) - \frac{1}{4} g(p) \leq \frac{3}{4}$$

s.t. $\frac{4}{3}\left(f(p) - \frac{1}{4} g(p)\right)$ is poly-bounded.

Lemma $\Rightarrow$ Thm

$f_1 = f \qquad g_1$ as in Lemma.

$$0 \leq \underbrace{\frac{4}{3}\left(f_1 - \frac{1}{4} g_1\right)}_{f_2} \leq 1$$

$f_2 \to g_2$ as in Lemma.

$$0 \leq \underbrace{\frac{4}{3}\left(f_2 - \frac{1}{4} g_2\right)}_{f_3} \leq 1$$

$f_3 \to g_3 \ldots$

$$f_1 = \frac{1}{4} g_1 + \frac{3}{4} f_2$$

$$f_2 = \frac{1}{4} g_1 + \frac{1}{4}\frac{3}{4} g_2 + \left(\frac{3}{4}\right)^2 f_3$$

$\ldots$

Geom

$$f_n = \sum_{k=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} g_k.$$

Sample index k w.p. $\frac{1}{4}\left(\frac{3}{4}\right)^{k-1}$

Sample from g_k. $\square$

Theorem (Keane, O'Brien)

Function $f: S \subseteq (0,1) \to (0,1)$ admits a factory if:

(1) it is continuous

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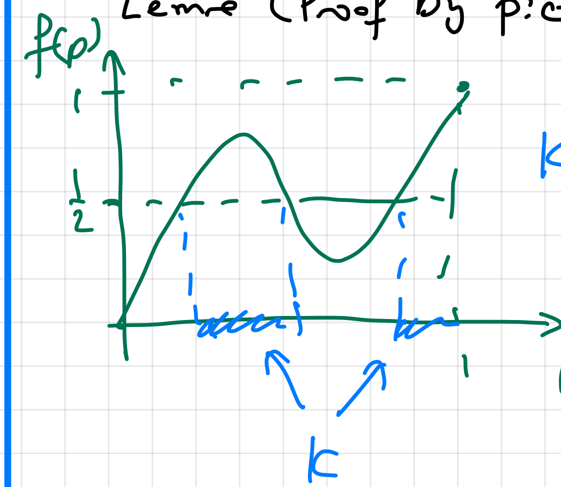
Lemma (Proof by picture)



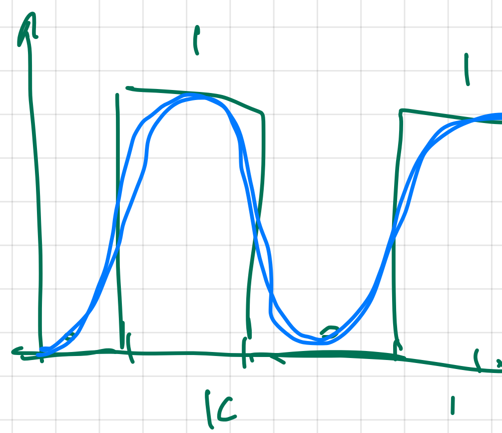
$K = \{p ; f(p) \geq \frac{1}{2}\}$

$X_i = \mathrm{Bernoulli}(p)$

$$g_m(p) \doteq \mathbb{P}\left(\frac{X_1 + \cdots + X_m}{m} \in K\right)$$

Lim $m \to \infty$ large enough m.

g_∞



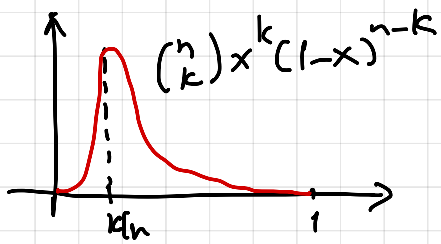
$\square$

Alternative Proof (Nacu-Peres)

① Bernstein approximation: given continuous function $f:[0,1]\to\mathbb{R}$

define $Q_n[f](x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$

then: $Q_n[f](x) \to f(x)$ uniformly.



② Polya's Thm: If $g(x,y)$ is a real homogeneous polynomial such that.
$g(x,y)>0 \;\forall x,y>0$ then $\exists n$ s.t all coefficients of $(x+y)^n g(x,y)$
are non-negative.

<u>Idea</u>: If $f:(0,1)\to[\varepsilon, 1-\varepsilon]$ use ① to construct approximators and
② to stitch the functions together in a series.

Fast Simulation

(Mossel-Peres 2002)
(Nacu-Peres 2005)

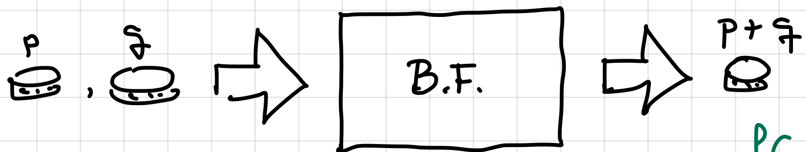
N = # coins tossed until output (random variable)

Property of f		Property of N	
continuous	$\Leftrightarrow$	$N < \infty$ a.s.	(i.e. it terminates)
real-analytic	$\Leftrightarrow$	$\mathbb{P}(N \geq n) \leq O(\rho^n)$	exponential tail. $\rho = \rho(p) < 1$.
Lipschitz	$\Leftarrow$??? $\Rightarrow$	$E[N] < \infty$	$f(p) = \sqrt{p}$? ? ?
rational	$\Leftrightarrow$	N via finite automata	

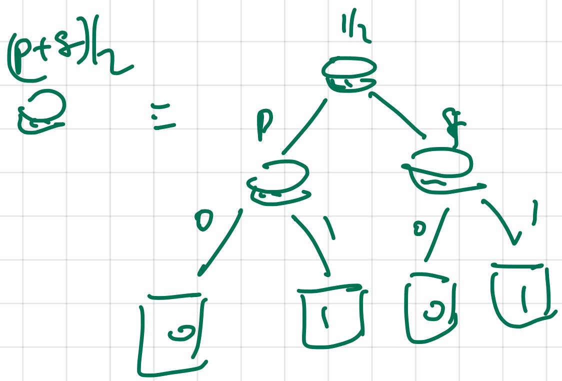
$e(p) := \mathbb{E}_p[N]$?

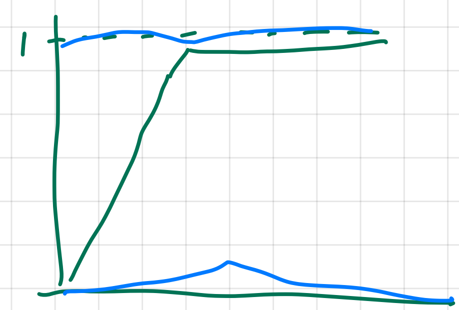
Problem: Sum of Two Coins

Given two coins p, q with the promise that $p+q \leq 1-\varepsilon$, sample from a $(p+q)$-biased coin.

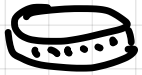
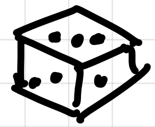


$f(p) = \min(2p, 1-\varepsilon)$

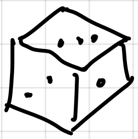




$$f\left(\frac{p+q}{2}\right) = p+q. \quad \square$$

Coins  to Dice 

(Mossel - Peres 2002)
(Dughmi et al 2017)
(Morina et al 2020)
↳ coupling from the past.
exact Markov chain samplers.

 $\longrightarrow$ 

$p_1, \quad p_2, \quad \ldots \quad p_n$

$$X = \begin{cases} 1 & \text{w.p. } p_1/(p_1+\cdots+p_n) \\ 2 & \text{w.p. } p_2/(p_1+\cdots+p_n) \\ \cdots & \\ n & \text{w.p. } p_n/(p_1+\cdots+p_n) \end{cases}$$

$$\mathbb{P}(X=i) = \frac{p_i}{n} + \underbrace{\left(1 - \sum_{j=1}^{n} \frac{p_j}{n}\right)}_{\text{restarts}} \mathbb{P}(X=i)$$

$X =$ [Sample index i uniformly
Flip coin p_i
If $\boxed{1}$ output $\boxed{i}$
Otherwise restart.]

$$\boxed{\mathbb{P}(X=i) = \frac{p_i}{\sum_{j=1}^{n} p_j}}$$ ✓

(Bernoulli Race).

Rational Functions

$0 \leq a(p) \leq b(p) \leq 1$

Factory for $f(p) = \dfrac{\overbrace{\sum_{i=0}^{k} a_i p^i}^{a(p)}}{\underbrace{\sum_{i=0}^{k} b_i p^i}_{b(p)}}$ $\quad f : (0,1) \to (0,1).$

① Bernoulli race between two coins $\overset{a(p)}{\underset{1}{\text{coin}}}$ and $\overset{b(p)-a(p)}{\underset{0}{\text{coin}}}$

Sample 1 w.p. $\dfrac{a(p)}{a(p) + (b(p) - a(p))} = \dfrac{a(p)}{b(p)}.$

Rational Functions

Factory for $f(p) = \dfrac{\sum_{i=0}^{k} a_i p^i}{\sum_{i=0}^{k} b_i p^i}$ $f: (0,1) \to (0,1)$.

② Polya's Thm: If $q(x,y)$ is a real homogeneous polynomial such that $q(x,y) > 0 \ \forall x,y > 0$ then $\exists n$ s.t. all coefficients of $(x+y)^n q(x,y)$ are non-negative.

$a(x) = \sum_{i=1}^{k} a_i x^i$ $A(x,y) = \sum_{i=1}^{k} a_i x^i (x+y)^{k-i}$ $a(p) = A(p, 1-p)$

$b(x) = \sum_{i=1}^{k} b_i x^i$ $B(x,y) = \sum_{i=1}^{k} b_i x^i (x+y)^{k-i}$ $b(p) = B(p, 1-p)$

$\dfrac{A(x,y)}{(x+y)^k} = \sum_{i=1}^{k} a_i \left(\dfrac{x}{x+y}\right)^i > 0$ $A(x,y) > 0 \quad B(x,y) > 0$

$B(x,y) - A(x,y) > 0$

$(x+y)^n A(x,y) = \sum_i c_i \, x^{a_i} y^{b_i}$ $c_i \geq 0$

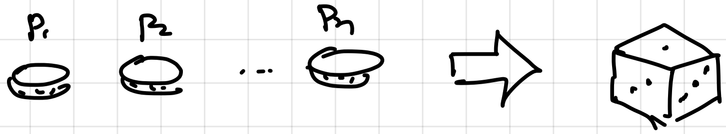
we can implement that!

replace $x = p$, $y = 1-p$ $a(p) = A(p, 1-p) = \sum_i c_i \, p^{a_i} (1-p)^{b_i}$

$\square$

Rational Functions

rational.

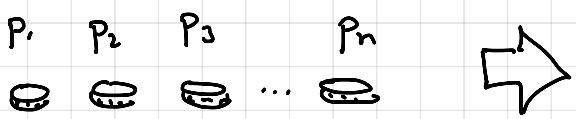


$$\mathbb{P}(X=i) = \frac{a_i(p_1,\dots,p_n)}{b_i(p_1,\dots,p_n)} \in (0,1).$$

Polya's Thm: If $g(x_1,\dots,x_m)$ is a real homogeneous polynomial such that $g(x_1,\dots,x_m) > 0$ for $x_i > 0 \ \forall i$ then $\exists n$ s.t. all coefficients of $(x_1+\dots+x_m)^n g(x_1,\dots,x_m)$ are positive.

Part II: Combinatorial Factories

(Niazadeh, PL, Schneider 2021)



random combinatorial object

Random Combinatorial Objects

(1) k- Subset

p_1 p_2 .. p_n promise $\sum p_i = k$.

Sample subset $S \subseteq [n]$ $|S| = k$

s.t. $\mathbb{P}[\, i \in S \,] = p_i$

$P = \{ x \in [0,1]^n ; \sum_{i=1}^{n} x_i = k \}.$

Vertices = permutations of $(\underbrace{1,1,..,1}_{k}\ \underbrace{0,0..0}_{n-k})$

(2) Matchings

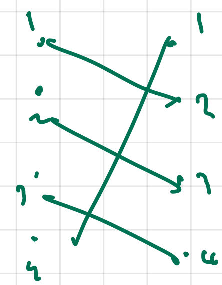
p_{ij}

$i=1..n$

$j=1..n$

promise : $\sum_i p_{ij} = 1 \ \forall j$
$\sum_j p_{ij} = 1 \ \forall i$

Sample matching M on $K_{n,n}$



$\mathbb{P}((i,j) \in M) = p_{ij}$

$$P = \left\{ p_{ij} \in [0,1]^{n \times n} ; \begin{array}{l} \sum_i p_{ij} = 1 \ \forall j \\ \sum_j p_{ij} = 1 \ \forall i \end{array} \right\}$$

Birkhoff- Von Neuman Polytope

Vertices of P = matchings

Random Combinatorial Objects

(3) spanning-trees

p_{ij}

$i \neq j$

$i=1..n$

$j=1..n$

$\sum_{ij} p_{ij} = n-1$

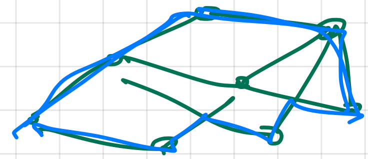
$\sum_{i,j \in S} p_{ij} \leq |S|-1, \quad \forall S \subseteq [n]$

sample tree T on K_n

$\mathbb{P}((i,j) \in T) = p_{ij}$

...

(4) s-t-flows / circulations

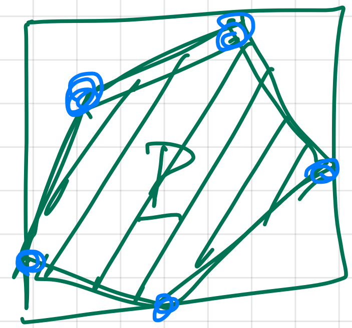


p_{ij}

...

Random Combinatorial Objects

(*) Vertices of a polytope



Polytope P contained in $[0,1]^n$

V = vertices of P

Given coins $\underset{p_1}{\ominus} \cdots \underset{p_n}{\ominus}$ s.t. $p = (p_1 \cdots p_n) \in P \cap (0,1)^n$

output a vertex $v \in V$ s.t. $\mathbb{E}[v_i] = p_i$

General Bernoulli Factory

Output set : V

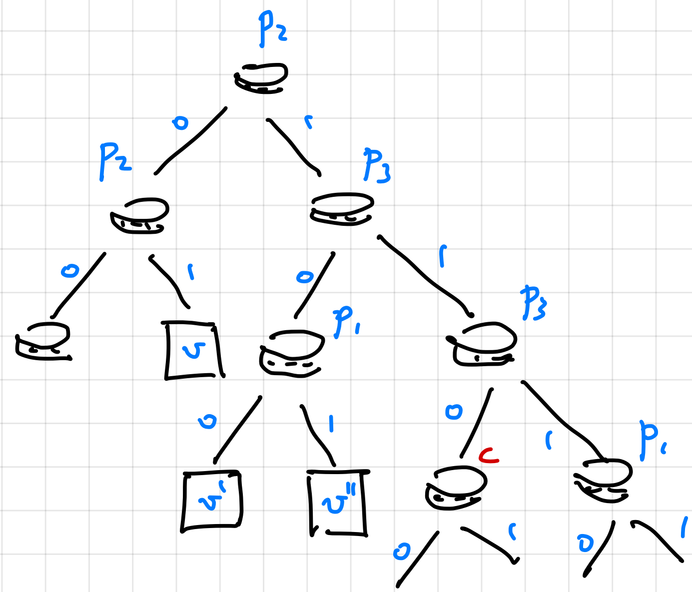
Input coins: p_1 p_2 … p_3

<u>Nodes:</u>

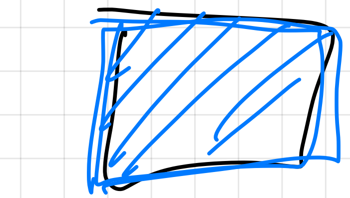
p_i - coin $\quad p_i$

helper coin $\quad c \quad$ (constant c)

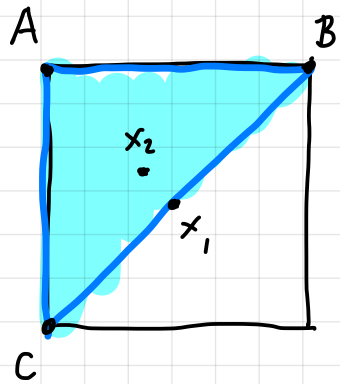
output node $\quad \boxed{v} \quad$ ($v \in V$)



Necessary Conditions



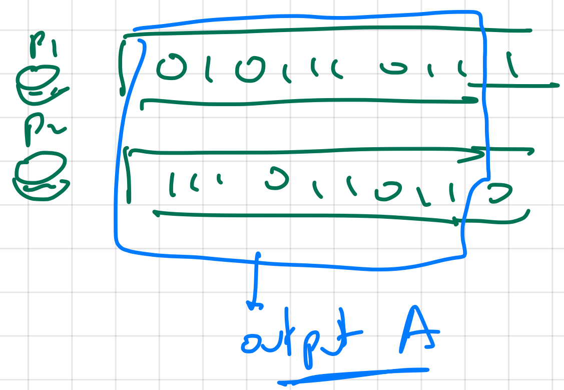
Claim: Impossible for Polytope below:

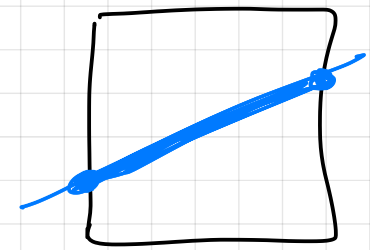


Output at x_1: B, C

Output at x_2: A, B, C

at x_2 $\mathbb{P}(\text{output } A) > 0$.





$\square$

Thm: Unless $P = [0,1]^n \cap$ affine subspace $\{p;\ Ap = b\}$ it is impossible to design a combinable factory...

Factory for k-Subset

Goal: sample 2-out-of-3

$1/3$ $\quad 2/3$ $\quad 1$

A $\quad$ B $\quad$ C

Natural Algorithms:

(1) Sample all coins: $S = \{ i : x_i = 1 \}$.

if $|S| = k$ output S

otherwise retry.

$$\frac{1}{3}(A,C) + \frac{2}{3}(B,C)$$

$$AC \sim \frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}$$

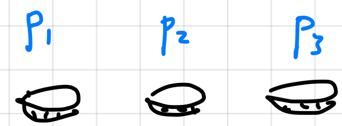
$$BC \sim \frac{2}{3} \cdot \frac{2}{3} \cdot 1 = \frac{4}{9}$$

(2) choose a pair (i,j) at random.

flip i j if both (1) output (i,j)

sometimes output (A,B)

Factory for k-Subset



Sampford Sampling

$$f_S(p) = \prod_{i\in S} p_i \prod_{i\notin S} (1-p_i) \cdot \sum_{j\in S} (1-x_j)$$

<u>Algorithm</u>

Flip each coin X_i

$S = \{i ; X_i = 1\}$

if $|S| \neq k$ restart

Chose a random coin j from S

Flip it again

If $\boxed{1}$ restart.

output S

$$\frac{\sum_{S;|S|=k} \mathbb{1}_S \cdot f_S(p)}{\sum_{S;|S|=k} f_S(p)} = p$$

$$\boxed{\sum_{v\in V} (v-p) f_S(p) = 0}$$

A Guess

For every polytope $P = \{x \in [0,1]^n ;\ Wx = b\}$ we can find polynomials $f_v(p_1 \cdots p_n)$ such that:

(1) $\sum_{v \in V} f_v(p_1 \cdots p_n) \cdot (p - v) = 0 \quad \forall p \in P.$

(2) $f_v(p_1 \cdots p_n)$ are implementable by a Bernoulli factory

$$f_v(p) = \sum_{i=0}^{k} c_i \, p_1^{a_i} (1-p_1)^{b_i} p_2^{c_i} (1-p_2)^{d_i} \cdots p_n^{e_i} (1-p_n)^{f_i} \cdots$$

Bernstein form

$c_i \geq 0$

Solve a Program

degree $\leq K$.

$$f_v(p) = \sum_{i=1} c_i \, p \ldots$$

$$\sum_v f_v(x_v) \cdot (x - v) = 0 \quad \forall p \ \text{ s.t. } Wp = b$$

$x_i \, x_j^2 \, x_k$

linear

$$E(c_1, c_2 \ldots c_n) = 0$$

$\Rightarrow$ LP

$c_i \geq 0$

(Manipulate polynomials)

(Solve linear programs)

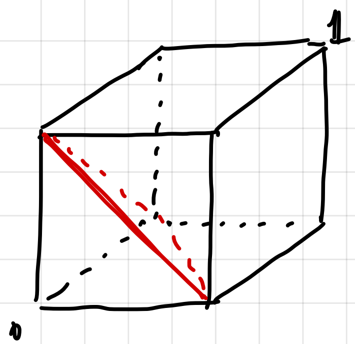
Next Simplest Example

Towards co-dimension 1: $P = \{x \in [0,1]^n ;\ w^T x = b\}$.

Simplest case we didn't understand: $\sum_i x_i = \alpha \qquad \alpha \notin \mathbb{Z}$.

Example: $n = 3 \qquad \alpha = 1.5$

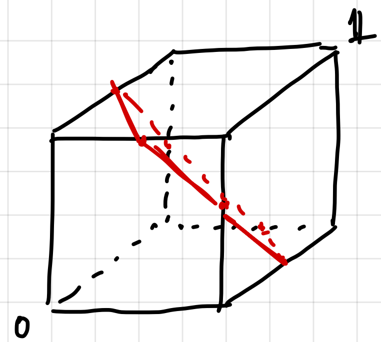
What are the vertices?



$\alpha = 1$

$|V| = 3$

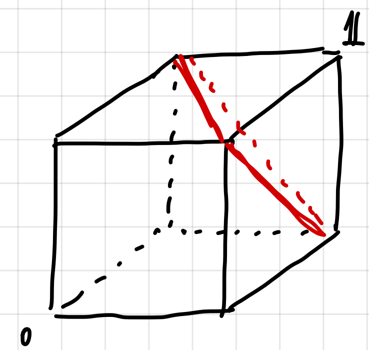
permutations of $(1,0,0)$



$\alpha = 1.5$

$|V| = 6$

permutations of $(1, \frac{1}{2}, 0)$



$\alpha = 2$

$|V| = 3$

permutations of $(1,1,0)$

Next Simplest Example

Towards co-dimension 1: $P = \{x \in [0,1]^n ;\ w^T x = b\}$.

Simplest case we didn't understand: $\sum_i x_i = \alpha$ $\quad \alpha \notin \mathbb{Z}$.

In general: n, $\alpha = k - \varepsilon$ $\quad 0 < \varepsilon < 1$

$V =$ permutations of $(\underbrace{1,1,\dots,1}_{k-1}, 1-\varepsilon, \underbrace{0,\dots 0}_{n-k})$

Start with $n = 3$ $\quad k = 1 + \varepsilon$ (simplest unknown example).

If polynomials exist we should be able to find it. After sterms at the solutions for very long:

$$f_v(p) = \underbrace{p_1 p_2 \cdots p_{k-1}}_{k-1}\ p_k (1-p_k) \underbrace{(1-p_{k+1}) \cdots (1-p_n)}$$

Co-dimension One

$P = \{ x \in [0,1]^n ;\ w^T x = b \}.$

Generic hyperplane avoiding $\{0,1\}^n$ $\Rightarrow$ Each vertex has one special index.

$v_i \in (0,1)$

$$f_v(p) = |w_i| \cdot p_i (1-p_i) \prod_{j : v_j = 1} p_j \prod_{j : v_j = 0} (1-p_j)$$

Non-generic hyperplanes:

Co-dimension One

$P = \{x \in [0,1]^n ;\ w^T x = b\}.$ $\qquad f_v(x) = |w_i|\ x_i(1-x_i) \prod_{j: v_j = 1} x_j, \prod_{j: v_j=0} (1-x_j)$

<u>Proof idea:</u> $\sum_v (v - x) f_v(x) = 0$

Vertices $v \in V \longleftrightarrow$ pairs (A, i) s.t. $\frac{b - w(A)}{w_i} \in (0,1)$ $\qquad w(A) = \sum_{j \in A} w_j$

$v = (A, i)$

Define $P_A(x) = \prod_{i \in A} x_i \prod_{i \notin A} (1 - x_i)$ s.t. $f_{(A,i)}(x) = |w_i| \cdot x_i\ P_A(x)$

Look at coordinate 1 of $\sum_{(A,i)} (v_1 - x_1) f_v(x) = 0$

$$\sum_{\substack{(A,i) \\ 1 \notin A \\ 1 \neq i}} (0 - x_i) f_{(A,i)}(x) + \sum_{\substack{(A,i) \\ 1 \in A}} (1 - x_i) f_{(A,i)}(x) + \sum_{\substack{(A,i) \\ i = A}} \left(\frac{b - w(A)}{w_i} - x_1 \right) f_{(A,i)}(x) = 0$$

Any dimension

$P(x) = \{ x \in \mathbb{R}^n ;\ W x = b \}$ W: $k \times n$ matrix of rank k.

Vertex v of P corresponds to a partition: (A, S, B)

$A = \{ i ;\ v_i = 0 \}$

$B = \{ i ;\ v_i = 1 \}$

$S = \{ i ;\ 0 < v_i < 1 \}$

$$W = \begin{array}{ccc} \overbrace{}^{A} & \overbrace{}^{S} & \overbrace{}^{B} \\ \left[\begin{array}{c|c|c} W_A & W_S & W_B \end{array}\right. & & \left.\right] \end{array}$$

$v_A = 0_A \qquad v_B = \mathbb{1}_B$

$v_S = W_S^{-1} (b - W_B \mathbb{1})$

Generic subspace $Ax = b$:

$$f_v(p) = \boxed{} \prod_{i \in A} p_i \prod_{i \in B} (1 - p_i) \prod_{i \in S} p_i (1 - p_i)$$

Factory for Matching

Matching is a bijection $\pi : [n] \to [n]$.

$$f_\pi(p) = \prod_{i=1}^{n} p_{i\pi(i)} \sum_{T \in Arb_1} \prod_{(u,v)\in T} x_{u,\pi(v)}$$

<u>Algorithm</u>:

Choose a random matching π.

Sample $p_{i,\pi(i)}$ $\forall i$. If any 0, restart.

Pick random spanning tree on K_n.

Orient edges of T towards 1.

For each $(i,j) \in T$, sample $p_{i,\pi(j)}$

If any 0, restart.

Output matching π.

Application to Mechanism Design

Mechanism Design Setup: n agents + space of outcomes X.

- allocations of items to agents
- flows/paths in a graph
- scheduling

Preference/type is a mapping: $v_i : X \to \mathbb{R}$. (private to agents).

Designer wants to optimize welfare $\sum_{i=1}^{n} v_i(x)$.

Two problems: (1) How to compute optimal allocation?

(2) How to incentivize agents to report truthfully?

Is there any reduction from (1) to (2)? Assume $A(v_1 \cdots v_n)$.

- Yes! If A is the optimal algorithm (Vickrey-Clarke-Groves 60s-70s)
- No, otherwise, without any prior about the valuations.

Application to Mechanism Design

However if $v_i \sim F_i$ iid, it is possible to come up with "BIC"-reductions:

- Yes! If types are single-parameter (Hartline - Lucier 2010)
- Yes*, in general (Hartline - Kleinberg - Malekian 2011)

  Algorithm $A \Rightarrow$ Mechanism M that is ε-BIC and

  $$\mathbb{E}[\text{welfare}(M)] \geq \mathbb{E}[\text{welfare}(A)] - \varepsilon.$$

  Technique: Replica - Surrogate Matching

- Yes, in general (Dughmi - Hartline - Kleinberg - Niazadeh 2017)

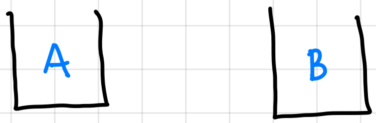
  Extra Ingredient is a Bernoulli Factory

Simple Mechanism Design Problem

- One agent with valuation $v: X \to \mathbb{R}$
- k urns, each with a distribution $F_1, \dots F_k$ on X.
- Designer has only sample access wants to maximize $\mathbb{E}_{x \sim F_i}[v(x)]$.

- <u>Example</u>: $X = \{1, 2, 3\}$ $\quad v(1) = 1 \quad v(2) = 2 \quad v(3) = 3$.

  Two urns $(k=2)$.



$\mathbb{P}(x=2) = 1$

$\mathbb{P}(x=1) = \frac{1}{2} + \varepsilon$

$\mathbb{P}(x=3) = \frac{1}{2} - \varepsilon$

- <u>Alternative</u>: Choose an urn with probability proportional to $\mathbb{E}_{x \sim F_j} \exp(\lambda (v(x) - 1))$

  + implicit payment computation (Babaioff, Kleinberg, Slivkins 2013)

Some Open Problems

(1) What are the conditions for $f:(0,1)^n \to (0,1)$ to admit a factory?

- how does the notion of poly-bounded generalize?

  $f(p) \geq \min(p_1, 1-p_1, p_2, 1-p_2, \dots, p_n, 1-p_n)^n$ for some n ?

- Nacu-Peres proof probably works for $f:(0,1)^n \to (\varepsilon, 1-\varepsilon)$.

(2) Remove "bounded-variation" from Keane-O'Brien proof.

Some Open Problems

(3) Dealing with $\{0,1\}$-values. E.g. extend results to $f:[0,1]\to[0,1]$.

E.g. Can we extend Sampford sampling to the boundary?

Not with an exponential tail.

$$P = \{x \in [0,1]^n ;\ \sum x_i = k\}.$$

$\mathbb{Z}$.

$$(1,1,0)$$

Some Open Problems

easy polytopes
"generic" ones:
every vertex has
k coordinates in $(0,1)^n$.

(4) Sample complexity: how many samples to obtain a matching or a k-subset? How optimized are the current factories with respect to $E[N]$?

For combinatorial factories $E[N] \sim \dfrac{n|V|}{\sum_v P_v(p)}$

$E[N] =$

$E[N] < \infty$.

(5) Explicit factories for other polytopes: e.g. flows / circulations.
Non-limit construction for generic polytopes.

non-generic polytopes.

↳ factories exist
how do they look like?

(6) Other applications to Game Theory / Mechanism Design
(e.g. Cai et al on revenue-preserving reductions)

Slides, Lecture Notes, References:

www.renatoppl.com/bernoulli

Thanks!