Feature-Based Dynamic Pricing

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[1]NYU Stern

[2]Google Research

Real estate agent problem

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3. Observes if the house was sold or not.
   - if $p_t \leq v(x_t)$, we sell and make profit p_t.
   - if $p_t > v(x_t)$, we don't sell and make zero profit.

Challenges and Assumptions

Learn/Earn or Explore/Exploit:
We don't know the market value $v(x_t)$.

Contextual problem:
The product is different in each round and adversarially chosen.

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3. Normalization: $\|x_t\| \leq 1, \forall t$, $\|\theta\| \leq R$.

Goal and Applications

Goal: Minimize worst-case regret.

$$\text{Regret} = \sum_{t=1}^{T} \theta^\top x_t - p_t \cdot \mathbf{1}\{p_t \leq \theta^\top x_t\}$$

Applications: online advertisement, real-estate, domain pricing, ...

Non-contextual setting

Simple setting:

One dimensional ($d = 1$) + no context $x_t = 1, \forall t$.

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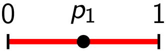
$K_0 =$ 

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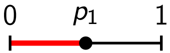
$K_1 =$ 

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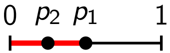
$K_1 =$ 

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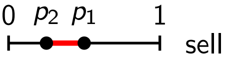
$K_2 =$ 

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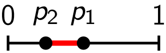
$K_2 =$ 

- after $\log(1/\epsilon)$ rounds we know $\theta \in [\hat{\theta}, \hat{\theta} + \epsilon]$.
- so $\hat{\theta}$ always sells so:

$$\text{Regret} \leq \log \frac{1}{\epsilon} + \left(T - \log \frac{1}{\epsilon} \right) \cdot \epsilon = O\left(\log T\right)$$

for $\epsilon = O(\log T / T)$.

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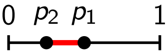
$K_2 =$ 

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- Leighton & Kleinberg: Optimal Regret $= O(\log \log T)$.

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Knowledge sets K_t

All θ compatible with observations so far.

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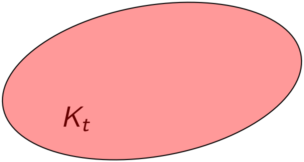

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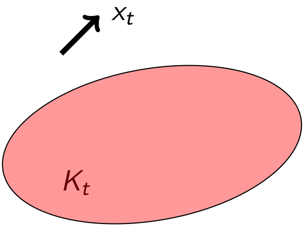

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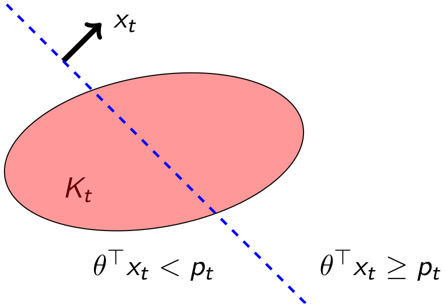

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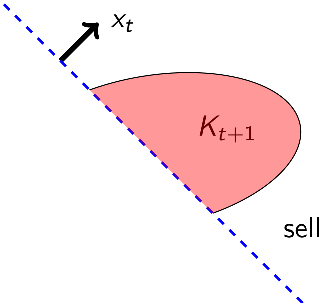

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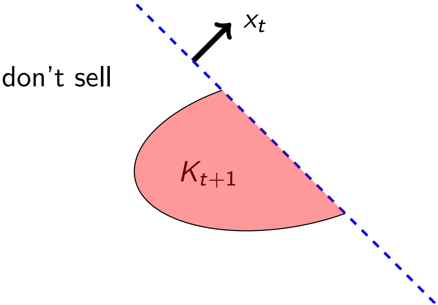

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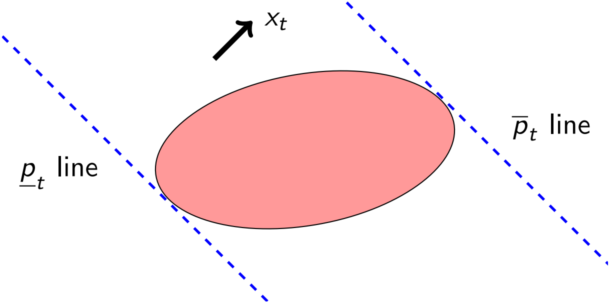
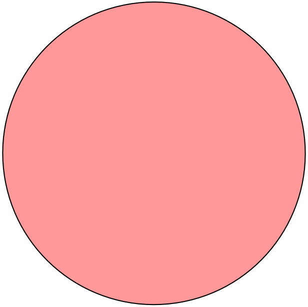
Price ranges $p_t \in [\underline{p}_t, \overline{p}_t]$

$\underline{p}_t = \min_{\theta \in K_t} \theta^\top x_t$ (exploit price, always sells)

$\overline{p}_t = \max_{\theta \in K_t} \theta^\top x_t$ (never sells)



Game: multi-dimensional binary search



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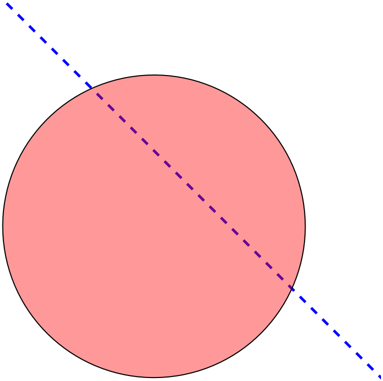

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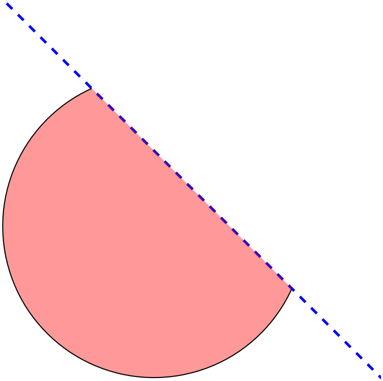

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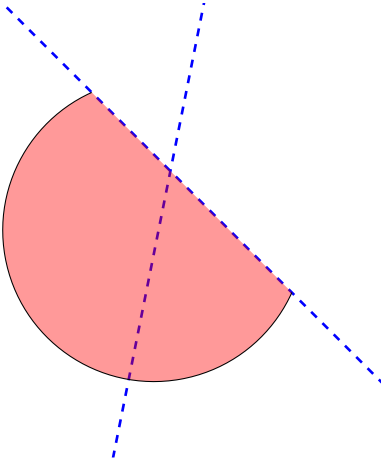

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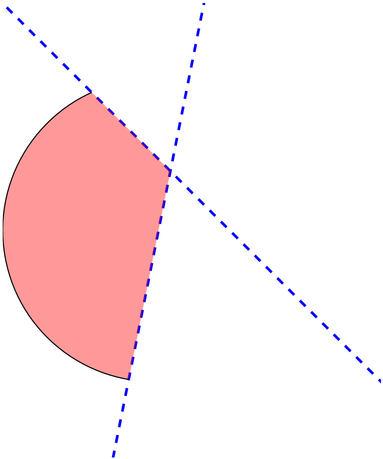

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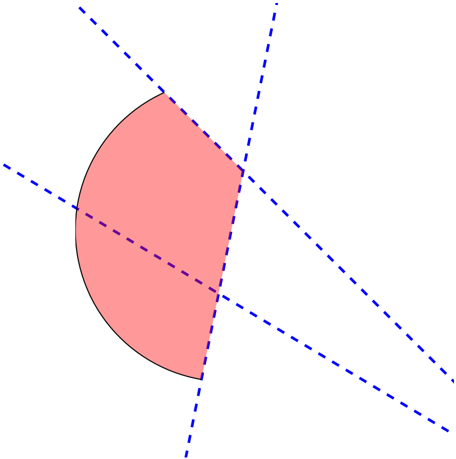

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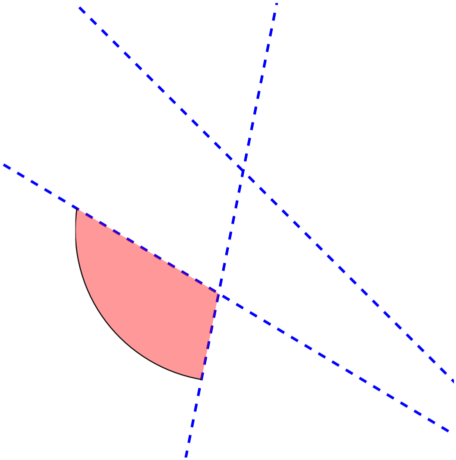

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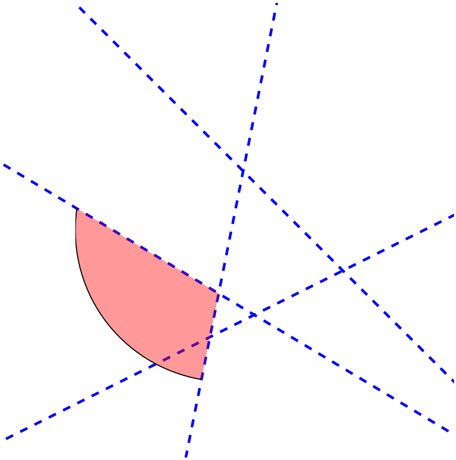

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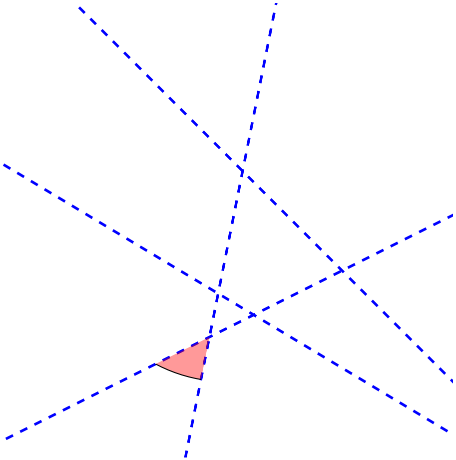

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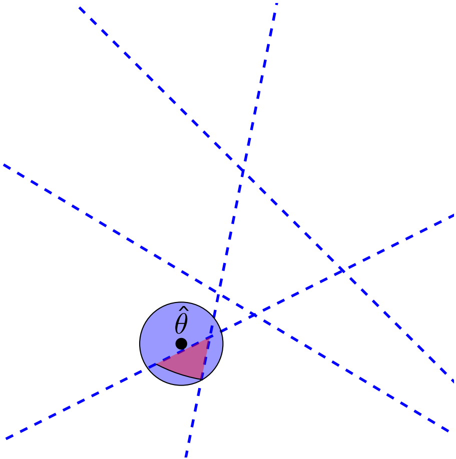

Our Goal: Find $\hat{\theta}$ such that $\|\theta - \hat{\theta}\| \leq \epsilon$, since $|\theta^\top x_t - \hat{\theta}^\top x_t| \leq \epsilon$ for all contexts x_t.

Idea # 1

Plan:

Narrow down K_t to $B(\hat{\theta}, \epsilon)$ and exploit from then on.

Issues with this approach:

- You may never see a certain feature.
- Some features might be correlated.
- Often it is not good to wait to profit.

Idea # 2

Plan:

Explore if there if enough uncertainty about $\theta^\top x_t$.
Compute $\overline{p}_t = \max_{\theta \in K_t} \theta^\top x_t$ and $\underline{p}_t = \min_{\theta \in K_t} \theta^\top x_t$
and exploit if

$$|\overline{p}_t - \underline{p}_t| \leq \epsilon$$

Which price to use in exploration:

From 1-dimensional binary search, we can try:

$$p_t = \frac{1}{2}\left(\overline{p}_t + \underline{p}_t\right)$$

Idea # 2

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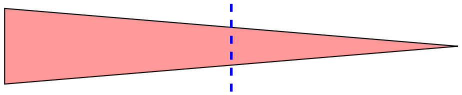
Thm: Regret of this approach is exponential in d.
Intuition: Shaving corners of a polytope in higher dimensions.



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Issues with this approach:

- Not easily computable.
- I don't know if it works or not.
- We get K_t of small volume: $\mathrm{vol}(K_t) \leq 2^{-t}$.
  What we want is $K_t \subseteq B(\hat{\theta}, \epsilon)$

Solution: Ellipsoids

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After cutting K_t regularize to its Löwner-John ellipsoid (same idea as in the Ellipsoid Method).

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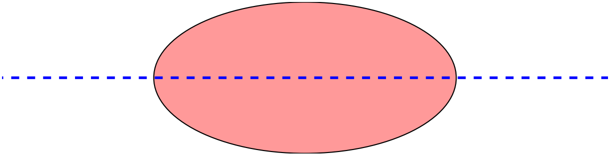

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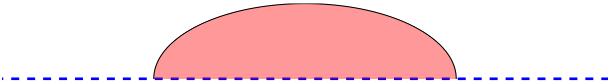

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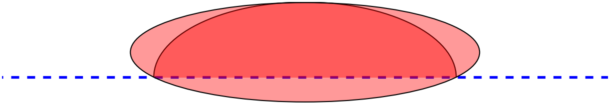

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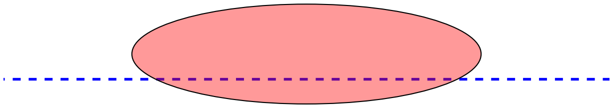

- We are keeping in the knowledge set some region that are known not to contain θ.
- Ellipsoids are simpler to control. We have a better grasp of them since they can be described by a simple formula:

$$E = \left\{\theta \in \mathbb{R}^d ; (\theta - \theta_0)^\top A^{-1} (\theta - \theta_0) \leq 1\right\}$$

for a positive definite matrix A.

Learning Algorithm

Initialize $A_0 = I/\sqrt{R}$ and $\theta_0 = 0$, i.e. $K_0 = B(0, R)$.
Implicitly we keep $K_t = \{\theta; (\theta - \theta_t)^\top A_t^{-1}(\theta - \theta_t) \leq 1\}$
For each timestep t:

- Receive feature vector $x_t \in \mathbb{R}^d$.
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$$A_{t+1} = \frac{d^2}{d^2+1}\left(A_t - \frac{2}{d+1} bb^\top\right)$$

and $\theta_{t+1} = \theta_t \pm \frac{1}{d+1} b$ where $b = -\theta_t + \mathrm{argmax}_{\theta \in K_t} \theta^\top x_t$.

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  (Solving a linear system since K_t is an ellipsoid)
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Main Theorem

Strategy for proving low regret

Guarantee a small number of exploration steps.

Lemma: If we explore for more than $O\left(Rd^2 \log\left(\frac{Rd}{\epsilon^2}\right)\right)$ steps, then K_t will be contained in a ball of radius ϵ. From then on, the algorithm will only exploit.

Theorem: Regret $\leq O(Rd^2 \log T)$ for $\epsilon = Rd^2/T$.

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- We know $\text{vol}_t = c_d \cdot \sqrt{\prod_i \lambda_i^t} = e^{-t/(d+1)}$. If we show that the smallest eigenvalue doesn't decrease too fast, then all the eigenvalues must eventually be small.
- We need to use the fact we never cut along directions that have small width, where width $= \overline{p}_t - \underline{p}_t$.

Controlling eigenvalues (high level details)

- Given eigenvalue of A_t we want to bound the eigenvalues of

$$A_{t+1} = \frac{d^2}{d^2+1} \underbrace{\left(A_t - \frac{2}{d+1} bb^\top\right)}_{B_{t+1}}$$

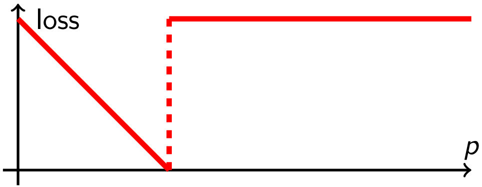
- If $\lambda_1^t \geq \ldots \geq \lambda_d^t$ are the eigenvalues of A_t, then the characteristic polynomial of B_{t+1} is:

$$\varphi_{B_{t+1}}(x) = \prod_j (\lambda_j - x) \cdot \underbrace{\left[1 - \beta \sum_i \frac{\tilde{b}_i^2}{\lambda_i - x}\right]}_{\hat{\varphi}_{B_{t+1}}}$$

- $\lambda_d^{t+1} \geq \lambda_d^t$ iff $\hat{\varphi}_{B_{t+1}}\left(\frac{d^2-1}{d^2}\lambda_d^t\right) \geq 0$. We show that this inequality holds whenever λ_d^t is small enough and $b^\top x \geq \epsilon$.

Connections

1. **Contextual Bandits:** We have a contextual bandit setting with adversarial context and a discontinuous loss function:



2. **Out of the shelf** contextual learning algorithms obtain $O(\sqrt{T})$ regret, are more computationally expensive, but don't assume that θ is fixed, instead they seek to be competitive against the best θ:

$$\text{Regret} = \max_{\theta} \sum_{t=1}^{T} \theta^{\top} x_t \cdot \mathbf{1}\{\theta^{\top} x_t \leq v_t\} - p_t \cdot \mathbf{1}\{p_t \leq v_t\}$$

3. **Quantum states (?):** Probing a buyer if he will buy at a certain price shares similarities with probing a quantum state with a linear measurement.

Lower bounds and Open Problems

1. A lower bound of $\Omega(d \log \log T)$ can be derived from embedding d independent instances of the 1-dimensional problem (feature vectors are coordinate vectors).
2. Other applications of multi-dimensional binary search.
3. Stochastic setting: $\theta \sim \mathcal{F}$, $x \sim \mathcal{D}$.

Thanks !