Polyhedral Clinching Auctions and the AdWords Polytope

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Creating an Ads campaign ...

Welcome to AdWords!

Create your first campaign

Getting started

1. Choose your budget
2. Create your ads
3. Select keywords that match your ads to potential customers
4. Enter your billing information.
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Creating an Ads campaign ...
How to deal with budgets in practice?

VCG, GSP, ...
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nice and well studied auction with good game-theoretic properties but without budgets...
How to deal with budgets in practice?

- budget layer
- VCG, GSP, ...
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- budget layer
- VCG, GSP, ...

engineering fix to adapt the original auction to the budgeted setting.

Original game theoretic analysis is now lost.
How to deal with budgets in practice?

- budget layer
- VCG, GSP, ...
How to deal with budgets in practice?

Goal:
Design an auction for AdWords that supports budgets natively, i.e., budgets are built in the game theoretic analysis.

Polyhedral Clinching Auction
What do we mean by budgets?
Classical quasi-linear utility function:

$$u_i = v_i(p_i) - p_i$$
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\[ u_i = v_i(\cdot) - p_i \]

Budget constrained utility function:

\[ u_i = v_i(\cdot) - p_i, \quad \text{if } p_i \leq B_i \]
\[ = -\infty, \quad \text{o.w.} \]
Classical quasi-linear utility function:

\[ u_i = v_i(p_i) - p_i \]

Very well understood: \textbf{VCG}, affine maximizers, ...

Budget constrained utility function:

\[
\begin{align*}
  u_i &= v_i(p_i) - p_i, & \text{if } p_i &\leq B_i \\
  &= -\infty, & \text{o.w.}
\end{align*}
\]
Classical quasi-linear utility function:

\[ u_i = v_i(b) - p_i \]

Very well understood: **VCG**, affine maximizers, ...

Budget constrained utility function:

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\[ = -\infty, \quad \text{o.w.} \]

Surprisingly little is known.
Goal: Design auctions for budget constrained agents
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- Agents report values and budgets \((v_i, B_i)\)
- Mechanism decides on allocation and payments for each player \((x_i, p_i)\)
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- Requirements:
  \[ p_i \leq B_i \]
  \[ (x_1, \ldots, x_n) \in P \]
Goal: Design auctions for budget constrained agents

• Agents report values and budgets \((v_i, B_i)\)

• Mechanism decides on allocation and payments for each player \((x_i, p_i)\)

• Requirements:
  \[ p_i \leq B_i \]
  \[ (x_1, \ldots, x_n) \in P \]
  (feasible set)
Desirable properties

- Incentive Compatibility:
  \[ v_i x_i(v, v^-) - p_i(v, v^-) \geq v_i x_i(v', v^-) - p_i(v', v^-) \]
  assumption: budgets \( B_i \) are public

- Individual rationality:
  \[ v_i x_i(v, v^-) - p_i(v, v^-) \geq 0 \]

- Pareto optimality:
  An outcome \((x, p)\) is **Pareto-optimal** if there is no \((x', p')\) such that \( u'_i \geq u_i, \sum p'_i \geq \sum p_i \) and at least one of them is strict.
Our main contribution

Solve this problem for a large class of feasible sets $\mathbf{P}$. 
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Solve this problem for a large class of feasible sets $\mathcal{P}$: (scaled) polymatroids.

Show this is impossible to be extended to general polytopes.
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Solve this problem for a large class of feasible sets $\mathbb{P}$: (scaled) polymatroids.

Show this is impossible to be extended to general polytopes.

Conjecture: scaled polymatroids are the largest class for which this is possible. (we supply evidence for that)
What do we know about budgets?

[Dobzinski, Lavi, Nisan, FOCS’08]
:: auction for one divisible good

[Fiat, Leonardi, Saia, Sankowski, EC’11]
:: auction for matching markets
What do we know about budgets?

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based on the **clinching auctions framework**

[Ausubel, AER’97]
How does it fit in our goal?

[Dobzinski, Lavi, Nisan, FOCS’08]
\( P = \{ x \in \mathbb{R}_+^n; \sum_i x_i \leq 1 \} \) Uniform Matroid

[Fiat, Leonardi, Saia, Sankowski, EC’11]
\( P = \text{Transversal Matroid} \)
How does it fit in our goal?

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\[ P = \{ x \in \mathbb{R}_+^n ; \sum_i x_i \leq 1 \} \text{ Uniform Matroid} \]

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For AdWords and other more complicated markets, we need to solve it for more generic feasibility constraints \( P \).
Our Results

We provide an auction with all the desirable properties for any polymatroid $\mathbf{P}$. 
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- Incentive compatibility
- Individual Rationality
- Budget Feasibility
- Pareto Optimality
Our Results

We provide an auction with all the desirable properties for any polymatroid $P$.

$$P = \{ x \in \mathbb{R}_+^n; \sum_{i \in S} x_i \leq f(S); \forall S \subseteq [n] \}$$

for a submodular function $f$. 
Our Results

We provide an auction with all the desirable properties for any polymatroid $\mathbf{P}$.

Our auction only needs oracle access to the submodular function $f$.

Our auction has a natural geometric flavor.
Our Results

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Many applications

Auctions for network design, queuing systems, video on demand, matching markets, internet advertisement, ...
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Our results

The set of \((x_1, \ldots, x_n)\) that can be obtained this way form a polymatroid. We call it the **AdWords Polytope**.

General model:
- multiple slots
- multiple keywords
- easy to generalize
Also on Sponsored Search with Budgets

Independently, [Colini-Baldeschi, Henzinger, Leonardi, Starnberger, 2012] design an auction for sponsored search with one keyword, multiple slots and budgets.
Our auction

polytope of feasible allocations

price clock
Our auction

\[ p = 0 \]
Our auction
Our auction
Our auction
Our auction

In each step compute demands $d_i$ at price $p$

$$d_i = \frac{B_{i}^{\text{rem}}}{p} \quad \text{if} \quad p \leq v_i \quad ; \quad \text{and} \quad 0 \quad \text{o.w.}$$
Our auction

In each step compute demands $d_i$ at price $p$

$$d_i = \frac{B_i^{\text{rem}}}{p} \quad \text{if} \quad p \leq v_i \ ; \text{and} \ 0 \ \text{o.w.}$$

Compute **clinched** amount $\delta_i$

$$\rho_i = \rho_i + \delta_i \quad B_i = B_i - p\delta_i$$
Computing clinched amounts $\delta_i$

What is the allocations that are still feasible at this point?
Our auction: how to implement clinch?

How much can I allocate to 1 without harming player 2?
Our auction: how to implement **clinch**?

\[
P^i_{\rho,d}(x_i) = \{ x_{-i} \in \mathbb{R}^{n-1}_+ \); (x_i, x_{-i}) \in P_{\rho,d} \}
\]

\[
P^i_{\rho,d}(x_i) \supseteq P^i_{\rho,d}(x'_i) \text{ if } x_i \leq x'_i.
\]
Our auction: how to implement *Clinch*?

\[
P^i_{\rho,d}(x_i) = \{ x_{-i} \in \mathbb{R}_+^{[n]\setminus{i}}; (x_i, x_{-i}) \in P_{\rho,d} \}
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P^i_{\rho,d}(x_i) \supseteq P^i_{\rho,d}(x'_i) \text{ if } x_i \leq x'_i.
\]

**Clinching step** \( \delta_i = \sup\{ x_i \geq 0; P^i_{\rho,d}(x_i) = P^i_{\rho,d}(0) \} \)
Our auction: how to implement clinch?

\[ P^i_{\rho,d}(x_i) = \{ x_{-i} \in \mathbb{R}^{n}_+ \setminus i; (x_i, x_{-i}) \in P_{\rho,d} \} \]

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**Clinching step** \[ \delta_i = \sup\{ x_i \geq 0; P^i_{\rho,d}(x_i) = P^i_{\rho,d}(0) \} \]

**Theorem:** Clinching as defined above results in a feasible allocation. If \( P \) is a polymatroid, \( \delta_i \) can be computed efficiently using submodular minimization.
Our auction: how to implement clinch?

\[ P^i_{\rho,d}(x_i) = \{ x_{-i} \in \mathbb{R}^n_+; (x_i, x_{-i}) \in P_{\rho,d} \} \]

\[ P^i_{\rho,d}(x_i) \supseteq P^i_{\rho,d}(x'_i) \text{ if } x_i \leq x'_i. \]

**Clinching step**

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**Theorem:** Clinching as defined above results in a feasible allocation. If \( P \) is a polymatroid, \( \delta_i \) can be computed efficiently using submodular minimization.

*[in practice there are more efficient algorithms for each case]*
Summary of the proof

• Show clinching is well-defined and can be computed efficiently

• Characterize Pareto-optimal outcomes for polymatroidal environments

• Show that the auction produces an outcome satisfying the characterization
Extensions and Limits

Going beyond polymatroids...
General convex environment
One budget-constrained player

For a single budget constrained player (and many other unconstrained ones), it is possible to design an auction for any convex environment.
What about 2 budget constrained players?

**Weak impossibility:** There is no auction following the clinching framework beyond (scaled) polymatroids.
What about 2 budget constrained players?

Weak impossibility: There is no auction following the clinching framework beyond (scaled) polymatroids.

Stronger impossibility: There exists a class of polytopes, for which **no auction exists** satisfying all the desirable properties.
What about 2 budget constrained players?

**Weak impossibility:** There is no auction following the clinching framework beyond (scaled) polymatroids.

**Stronger impossibility:** There exists a class of polytopes, for which no auction exists satisfying all the desirable properties.

No hope of an auction for a general polyhedral environment.
Impossibility for decreasing marginals

Single divisible good: \( \{ x \in \mathbb{R}_+^n; \sum_i x_i \leq 1 \} \)

Decreasing marginal valuations
Impossibility for decreasing marginals

Single divisible good: \( \{ x \in \mathbb{R}_+^n; \sum_i x_i \leq 1 \} \)

**Thm:** No auction with all the desirable properties for one divisible good with decreasing marginals.

Strengthens previous impossibility results of [Lavi, May’11] and [Fiat et al’11]
Summary

Clinching auction for polymatroids

\[ P_{\rho,d}(0) \quad P_{\rho,d}(\delta_1) \quad P_{\rho,d}(\delta_1 + \epsilon) \]
Summary

Clinching auction for polymatroids

Characterization of Pareto Optimal Auctions in general polyhedral environments
Summary

Clinching auction for polymatroids

Characterization of Pareto Optimal Auctions in general polyhedral environments

Impossibility for decreasing-marginals and budgets

Impossibility for general polytopes
Summary

Clinching auction for polymatroids

Positive results for one budget-constrained agent and general environments

Impossibility for general polytopes

Characterization of Pareto Optimal Auctions in general polyhedral environments

Impossibility for decreasing-marginals and budgets
Thanks !